King Fahd University of Petroleum and Minerals

Department of Mathematics and Statistics

Math 302 Exam II

Semester (141) Nov. 20, 2014 at 5:45-7:45 PM

Name: ....Solution........

I.D: .... Section: .... Serial: ....

Question	Points
1	/20
2	/20
3	/20
4	/20
5	/20
Total	/100

Question 1 (20 points)

Consider $\mathbf{r}_1(t) = \langle \frac{1}{3}t^3, t^2, 2t\rangle$ and $\mathbf{r}_2(t) = \langle e^t, t, \sin t\rangle$.

a. Find the length of the curve C defined by $\mathbf{r}_1(t)$, $0 \le t \le 2$.

b. Find $\frac{d}{dt}(\mathbf{r}_1(t) \times \mathbf{r}_2(t))$.

Sol.

a. length $\ell = \int_0^2 \|r'(t)\|\, dt = \int_0^2 \sqrt{f'^2 + g'^2 + h'^2}\, dt$

$$= \int_0^2 \sqrt{t^4 + 4t^2 + 4}\, dt = \int_0^2 \sqrt{(t^2+2)^2}\, dt$$

$$= \int_0^2 (t^2+2)\, dt = \frac{t^3}{3} + 2t \Big]_0^2 = \frac{20}{3}.$$

b. $\frac{d}{dt}\left(r_1(t) \times r_2(t)\right) = r_1'(t) \times r_2(t) + r_1(t) \times r_2'(t)$

$$= \begin{vmatrix} i & j & k \\ t^2 & 2t & 2 \\ e^t & t & \sin t \end{vmatrix} + \begin{vmatrix} i & j & k \\ \frac{1}{3}t^3 & t^2 & 2t \\ e^t & 1 & \cos t \end{vmatrix}$$

$$= (t^2\cos t - 4t + 2t\sin t)\,\vec{i}$$
$$-\left(\frac{1}{3}t^3\cos t - 2te^t - 2e^t + t^2\sin t\right)\vec{j}$$
$$+\left(\frac{4}{3}t^3 - 2te^t - t^2e^t\right)\vec{k}.$$

Question 2 (20 points)

a. Let the surface S be defined by $z = \ln(x^2 + y^2)$ and the point $\mathbf{P}(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$. Find the equations of the ***tangent plane*** and the ***normal line*** to the surface S at $\mathbf{P}$.

b. If $f(x,y) = x^2 + xy + y^2 - x$, find all points where $D_{\mathbf{u}}f(x,y)$ in the direction of $\mathbf{u} = \mathbf{i} + \mathbf{j}$ is zero.

Sol.

a. S is given by $\phi(x,y,z) = z - \ln(x^2+y^2)$

Clearly, $P = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0) \in S$, since $0 = \ln(\frac{1}{2} + \frac{1}{2})$.

$$\nabla\phi = \left\langle \frac{-2x}{x^2+y^2}, \frac{-2y}{x^2+y^2}, 1 \right\rangle$$

$$\nabla\phi(P) = \langle -\sqrt{2}, -\sqrt{2}, 1 \rangle$$

Tangent plane: $-\sqrt{2}\left(x - \frac{1}{\sqrt{2}}\right) - \sqrt{2}\left(x - \frac{1}{\sqrt{2}}\right) + (z - 0) = 0$

$\Leftrightarrow \quad -\sqrt{2}\,x - \sqrt{2}\,y + z + 2 = 0$, or

$$\boxed{\sqrt{2}\,x + \sqrt{2}\,y - z = 2}$$

Normal lines

parametric equations

$x = \frac{1}{\sqrt{2}} - \sqrt{2}\,t$

$y = \frac{1}{\sqrt{2}} - \sqrt{2}\,t$

$z = t$

symmetric equations

$$\frac{x - \frac{1}{\sqrt{2}}}{-\sqrt{2}} = \frac{y - \frac{1}{\sqrt{2}}}{-\sqrt{2}} = \frac{z}{1}$$

or

$$\frac{\sqrt{2}\,x - 1}{2} = \frac{\sqrt{2}\,y - 1}{2} = -z$$

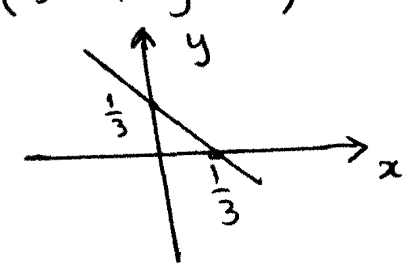
b. $\nabla f(x,y) = \langle 2x + y - 1, x + 2y \rangle$

The unit vector $v = \frac{u}{\|u\|} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$

Directional derivative: $D_u f(x,y) = \frac{1}{\sqrt{2}}(3x + 3y - 1)$.

$D_u f(x,y) = 0 \Leftrightarrow x + y = \frac{1}{3}$.

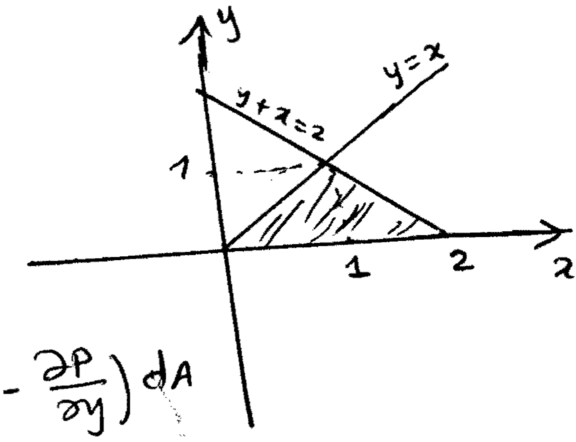
The set of all points is the line $x + y = \frac{1}{3}$



Question 3 (20 points)

Find $\oint_C (y^2 + e^{-x^2})dx + (xy + 2)dy$, where C is the positively oriented closed triangle bounded by the lines $y = x$, $x + y = 2$ and $x - axis$.

Sol.



$$P = y^2 + e^{-x^2} \Rightarrow \frac{\partial P}{\partial y} = 2y$$

$$\varphi = xy + 2 \Rightarrow \frac{\partial \varphi}{\partial x} = y$$

So $$I = \oint_C (P\,dx + \varphi\,dy) = \iint_R \left(\frac{\partial \varphi}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_R -y\,dA$$

$$R = \{(x,y) / \ 0 \le y \le 1, \quad y \le x \le 2-y\}$$

$$= \{(x,y) / \ 0 \le x \le 1, \quad 0 \le y \le x\} \cup \{(x,y) / 1 \le x \le 2, 0 \le y \le 2-x\}$$

Therefore $$I = \int_0^1 \int_y^{2-y} -y\,dx\,dy = -\int_0^1 (2y - 2y^2)\,dy$$

$$= -\left[y^2 - \frac{2}{3}y^3\right]_0^1 = -1 + \frac{2}{3} = -\frac{1}{3}$$

or

$$I = \int_0^1 \int_0^x -y\,dy\,dx + \int_1^2 \int_0^{2-x} -y\,dy\,dx$$

$$= -\frac{1}{6} - \frac{1}{6} = -\frac{1}{3}.$$

Question 4 (20 points)

The force $\mathbf{F}(x,y,z) = 2xz\mathbf{i} + 2yz\mathbf{j} + (x^2+y^2+1)\mathbf{k}$ is moving an object on a curve given by $C = \{\mathbf{r}(t) = \langle e^{t^2}, t, \sin t\rangle,\ 0 \le t \le \pi\ \}$.

a. Check if **F** is conservative.

b. Find the work $W = \int_C \mathbf{F}.d\mathbf{r}$ done by the force **F**.

Sol.

a. $\text{Curl}\,F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ 2xz & 2yz & x^2+y^2+1 \end{vmatrix}$

$= (2y-2y)\vec{i} - (2x-2x)\vec{j} + (0-0)\vec{k} = \vec{0}$

So F is conservative since it has continuous partial derivatives and Curl $F = \vec{0}$.

b. let ϕ be a potential of F. Then $\nabla\phi = F$.

$\phi_x = P = 2xz \Rightarrow \phi = x^2 z + \underline{\psi(x,y)}$

$\phi_y = \psi_y = Q = 2yz \Rightarrow \psi(x,y) = y^2 z + \underline{\alpha(z)}$

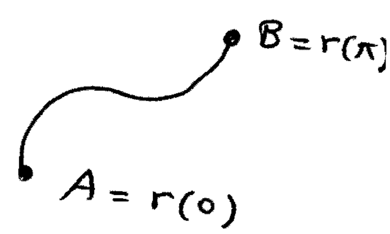
$\Rightarrow \phi = x^2 z + y^2 z + \alpha(z)$

$\Rightarrow \phi_z = x^2 + y^2 + \alpha'(z) = x^2+y^2+1 = R$

$\Rightarrow \alpha'(z) = 1 \Rightarrow \alpha = z + c$

So a potential $\phi(x,y,z) = x^2 z + y^2 z + z$ (Take $c=0$)

The work is



$W = \int_C F.dr = \phi(B) - \phi(A)$

$= \phi(r(\pi)) - \phi(r(0))$

$= \phi(e^{\pi^2}, \pi, 0) - \phi(1,0,0) = 0 - 0 = 0.$

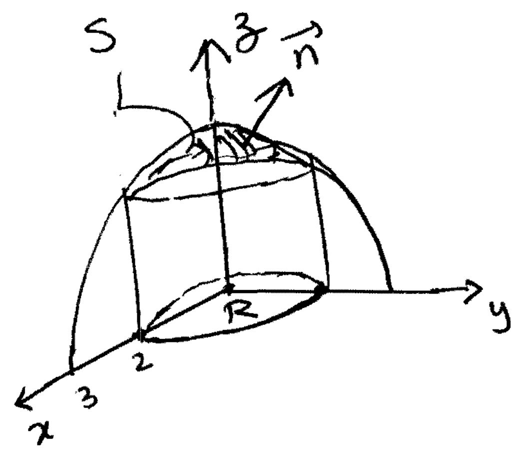
Notice that C is not closed.

Question 5 (20 points)

Use Stokes' theorem to evaluate $\oint_C$ **F**$.d$**r**, where $\mathbf{F}(x,y,z) = \sin(x^2)\,\mathbf{i} - z\mathbf{j} + y\mathbf{k}$ and the curve C is the trace of the cylinder $x^2 + y^2 = 4$ in the parapoloid $z = 9 - x^2 - y^2$ [redacted]. Assume that C is oriented counter clock-wise as viewed from above.

Sol.

$$\text{Curl } F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin(x^2) & -z & y \end{vmatrix}$$



$$= (1+1)\vec{i} - (0-0)\vec{j} + (0-0)\vec{k}$$

$$= 2\vec{i}$$

Stokes theorem $\oint_C F.dr = \iint_S \text{Curl} F . \eta \, ds$

The surface S is given by $z + x^2 + y^2 = 9 = \phi(x,y,z)$.

$\Rightarrow \nabla\phi = \langle 2x, 2y, 1\rangle \Rightarrow |\nabla\phi| = \sqrt{1+4(x^2+y^2)}$

Thus $\eta = \frac{1}{\sqrt{1+4(x^2+y^2)}} \langle 2x, 2y, 1\rangle$.

$$\Rightarrow I = \oint_C F.dr = \iint_S \text{Curl } F . \eta \, ds = \iint_S \frac{2x}{\sqrt{1+4(x^2+y^2)}} \, dS$$

$$= \iint_R \frac{4x}{\cancel{\sqrt{1+4(x^2+y^2)}}} \cancel{\sqrt{1+f_x^2+f_y^2}} \, dA$$

where R is the disk Centered at $(0,0,0)$ with radius $= 2$

and $z = f(x,y) = 9 - x^2 - y^2$

$$\therefore \quad I = \int_0^{2\pi}\int_0^2 \frac{4r\cos\theta}{\cancel{\sqrt{1+4r^2}}} \cancel{\sqrt{1+4r^2}} \, r\,dr\,d\theta = 0$$