King Fahd University of Petroleum & Minerals
Department of Mathematics and Statistics
MATH 301 – Methods of Applied Mathematics
FINAL EXAM
2014-2015 (141)

Tuesday, December 30, 2014 **Allowed Time: 150 min.**

ANSWER KEY

Instructions:

1. Write neatly and legibly. You may lose points for messy work.

2. Show all your work for written questions. No points for answers without justification.

3. Calculators and Mobiles are not allowed.

4. Make sure that you have 11 different problems (6 pages + cover page).

WRITE ANSWERS TO MCQ HERE:

MCQ	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8
ANSWER	A	A	A	A	A	A	A	A

Problem No.	Points	Maximum Pts
MCQ		64
9		28
10		22
11		26
Total		140

Q1 (8 pts): If $f(t) = \mathcal{L}^{-1}\left\{\frac{2s}{s^2-4s+3}\right\}$, then $f(0)$ is equal to

(A) 2 (B) 1 (C) -2 (D) 4 (E) 0

Q2 (8 pts): If $f(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+2s+2}\right\}$, then $f(\frac{\pi}{4})$ is equal to

(A) 0 (B) $\sqrt{2}e^{-\frac{\pi}{4}}$ (C) $\frac{1}{\sqrt{2}}e^{-\frac{\pi}{4}}$ (D) $\frac{1}{\sqrt{2}}e^{\frac{\pi}{4}}$ (E) $e^{-\pi}$

Q3 (8 pts): If $F(s) = \mathcal{L}\left\{\int_0^t e^{3\tau}\cos 2(t-\tau)\,d\tau\right\}$, then $F(4)$ is equal to

(A) $\frac{1}{5}$ (B) $\frac{4}{17}$ (C) $\frac{4}{51}$ (D) $\frac{1}{10}$ (E) $\frac{1}{15}$

<u>Q4 (8 pts)</u>: If $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\pi x + b_n \sin n\pi x)$

is the Fourier series of $f(x) = x + 1$ on the interval $[-1,1]$, then $a_0 + a_1 + b_2$ is equal to

(A) $\frac{2\pi-1}{\pi}$ (B) $\frac{\pi-1}{\pi}$ (C) $\frac{2\pi^2-2}{\pi^2}$ (D) $\frac{2\pi+1}{\pi}$ (E) $\frac{2\pi-2}{\pi}$

<u>Q5 (8 pts)</u>: If the solution of the BVP $\begin{cases} 4u_{xx} = u_{tt}, & 0 < x < 1, \ t > 0 \\ u(0,t) = 0, \ u(1,t) = 0, & t > 0 \\ u(x,0) = 0, \ u_t(x,0) = 9, & 0 < x < 1 \end{cases}$

is given by $u(x,t) = \sum_{n=1}^{\infty} B_n \sin 2n\pi t \sin n\pi x$, then the value of B_3 is equal to

(A) $\frac{2}{\pi^2}$ (B) $\frac{12}{\pi}$ (C) $\frac{1}{\pi^2}$ (D) $\frac{6}{\pi}$ (E) $\frac{6}{\pi^2}$

<u>Q6 (8 pts)</u>: If $\sum_{n=0}^{\infty} C_n P_n(x)$ is the Legendre series expansion of $f = \begin{cases} 0, & -1 < x < 0 \\ 4x^2, & 0 \le x < 1 \end{cases}$

then the value of C_2 is equal to

(A) $\frac{4}{3}$ (B) $\frac{5}{2}$ (C) $\frac{10}{3}$ (D) $\frac{5}{4}$ (E) $\frac{8}{3}$

Q7 (8 pts): Consider the regular Sturm-Liouville problem

$$x^2y'' + 5xy' + \lambda y = 0, \qquad y(1) = 0, y(2) = 0$$

and let y_n and y_m be two eigenfunctions corresponding to two different eigenvalues. Which of the following is **true**?

(A) $\int_1^2 x^3 y_n(x) y_m(x) dx = 0$

(B) $\int_1^2 y_n(x) y_m(x) dx = 0$

(C) $\int_1^2 x y_n(x) y_m(x) dx = 0$

(D) $\int_1^2 \frac{1}{x^2} y_n(x) y_m(x) dx = 0$

(E) $\int_1^2 x^2 y_n(x) y_m(x) dx = 0$

Q8 (8 pts): Consider the following Laplace's equation in cylindrical coordinates

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0, \qquad 0 < r < 1, 0 < z < 1$$

Using the separation of variables $u(r,z) = R(r)Z(z)$, with a separation constant $-\alpha^2$, a bounded product solution of this PDE is given by

(A) $u(r,t) = (c_1 \cosh \alpha z + c_2 \sinh \alpha z) J_0(\alpha r)$

(B) $u(r,t) = c e^{\alpha^2 z} J_0(\alpha r)$

(C) $u(r,t) = (c_1 \cos \alpha z + c_2 \sin \alpha z) J_0(\alpha r)$

(D) $u(r,t) = r^\alpha (c_1 \cos \alpha z + c_2 \sin \alpha z)$

(E) $u(r,t) = r^\alpha (c_1 \cosh \alpha z + c_2 \sinh \alpha z)$

Q9 (28 pts): Use the method of **separation of variables** to solve the heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad 0 < x < \pi, \;\; t > 0$$

Subject to $\qquad u(0,t) = 0, \;\; u(\pi,t) = 0, \qquad t > 0$

$$u(x,0) = f(x), \qquad 0 < x < \pi$$

Solution:

3 Let $u(x,t) = X(x)T(t) \underset{\text{in the PDE}}{\Longrightarrow} X^{''}T = XT^{'} \Rightarrow \frac{X^{''}}{X} = \frac{T^{'}}{T} = -\lambda$

1 • $u(0,t) = 0 \Rightarrow X(0)T(t) = 0 \Rightarrow X(0) = 0$

1 • $u(\pi,t) = 0 \Rightarrow X(\pi)T(t) = 0 \Rightarrow X(\pi) = 0$

3 (1) $\frac{T'}{T} = -\lambda \Rightarrow \cdots \Rightarrow T(t) = ce^{-\lambda t}$

1 (2) $\frac{X''}{X} = -\lambda \Rightarrow X'' + \lambda X = 0, \qquad X(0) = 0$ and $X(\pi) = 0$

1 Solving (2): $\qquad m^2 + \lambda = 0 \Rightarrow m = \pm\sqrt{-\lambda}$

1 ◙ Case I: $-\lambda = 0 \Rightarrow m = 0,0 \Rightarrow X(x) = c_1 + c_2 x$

1 • $X(0) = 0 \Rightarrow c_1 = 0 \Rightarrow X(x) = c_2 x$

1 • $X(\pi) = 0 \Rightarrow c_2\pi = 0 \Rightarrow c_2 = 0 \Rightarrow X(x) \equiv 0$ the trivial solution

◙ Case II: $-\lambda > 0$, so we put $-\lambda = \alpha^2$ $(\alpha > 0) \Rightarrow m = \pm\sqrt{\alpha^2} = \alpha, -\alpha$

1 $$\Rightarrow X(x) = c_1 \cosh \alpha x + c_2 \sinh \alpha x$$

1 • $X(0) = 0 \Rightarrow c_1 = 0 \Rightarrow X(x) = c_2 \sinh \alpha x$

1 • $X(\pi) = 0 \Rightarrow c_2 \sinh \alpha\pi = 0 \Rightarrow c_2 = 0 \Rightarrow X(x) \equiv 0$ the trivial solution

◙ Case III: $-\lambda < 0$, so we put $-\lambda = -\alpha^2$ $(\alpha > 0) \Rightarrow m = \pm\sqrt{-\alpha^2} = \pm\alpha i$

1 $$\Rightarrow X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$$

1 • $X(0) = 0 \Rightarrow c_1 = 0 \Rightarrow X(x) = c_2 \sin \alpha x$

• $X(\pi) = 0 \Rightarrow c_2 \sin \alpha\pi = 0 \Rightarrow$ we take $c_2 \neq 0$ and $\sin \alpha\pi = 0$

2 $$\Rightarrow \alpha\pi = n\pi \Rightarrow \alpha = n, \qquad n = 1,2,3,\ldots$$

$\Longrightarrow$ the eigenvalues are $\lambda_n = n^2$ and the nontrivial solutions are

1 $$X_n(x) = c_2 \sin nx$$

1 Now, we use $\lambda_n = n^2$ in (1) to get $T_n(t) = ce^{-n^2 t}$

So, product solutions are

1 $$u_n(x,t) = X_n T_n = (c_2 \sin nx)\left(ce^{-n^2 t}\right) = A_n e^{-n^2 t} \sin nx, \qquad n = 1,2,3, \ldots$$

Then, by superposition principle, the general solution is

2 $$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-n^2 t} \sin nx$$

Using the initial condition $u(x,0) = f(x)$, we find that

$$f(x) = \sum_{n=1}^{\infty} A_n \sin nx$$

which is the sine series of $f(x)$. Therefore

3 $$A_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

Hence, the solution is: $$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \right) e^{-n^2 t} \sin nx$$

Q10 (22 pts): Use the **Laplace transform** to solve the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \qquad x > 0, \ \ t > 0$$

Subject to $\quad u_x(0,t) = t^2, \quad \lim_{x\to\infty} u(x,t) = 0, \quad t > 0$

$$u(x,0) = 0, \ \ u_t(x,0) = 0, \quad x > 0$$

Solution: We take the Laplace transform with respect to the variable t

4 $$\mathcal{L}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \mathcal{L}\left\{\frac{\partial^2 u}{\partial t^2}\right\} \Longrightarrow \frac{\partial^2 U(x,s)}{\partial x^2} = s^2 U(x,s) - su(x,0) - u_t(x,0)$$

1 $$\Longrightarrow \frac{\partial^2 U(x,s)}{\partial x^2} - s^2 U(x,s) = 0$$

3 The auxiliary equation is: $m^2 - s^2 = 0 \Longrightarrow m = \pm s \Longrightarrow U(x,s) = c_1 e^{-sx} + c_2 e^{sx}$

Now, we take the Laplace transform of the remaining conditions

2 (a) $\mathcal{L}\left\{\lim_{x\to\infty} u(x,t)\right\} = \mathcal{L}\{0\} \Longrightarrow \lim_{x\to\infty} U(x,s) = 0$

2 (b) $\mathcal{L}\{u_x(0,t)\} = \mathcal{L}\{t^2\} \Longrightarrow U_x(0,s) = \frac{2}{s^3}$

2 • Using (a) gives: $\quad c_2 = 0 \Longrightarrow U(x,s) = c_1 e^{-sx}$

• Using (b) gives: $\quad -c_1 s = \frac{2}{s^3} \Longrightarrow c_1 = -\frac{2}{s^4}$

2 $$\therefore U(x,s) = -\frac{2}{s^4} e^{-sx}$$

4 $$\Longrightarrow u(x,t) = \mathcal{L}^{-1}\left\{-\frac{2}{s^4} e^{-sx}\right\} = -\frac{1}{3}(t-x)^3 \mathcal{U}(t-x)$$

2 $$= \begin{cases} 0, & 0 \le t < x \\ -\frac{1}{3}(t-x)^3, & t \ge x \end{cases}$$

Q11 (26 pts): The steady-state temperature in a semi-infinite plate is modeled by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad x > 0,\ \ 0 < y < \pi$$

$$u_x(0, y) = \sin y\,, \qquad 0 < y < \pi$$

$$u(x, 0) = 0,\ \ u(x, \pi) = f(x) = \begin{cases} 1, & 0 < x \le 1 \\ 0, & x > 1 \end{cases}$$

Use either the **Fourier sine or cosine transform** to solve this IBVP.

1 **Solution**: Using Fourier cosine transform with respect to the variable x

4 $$\mathcal{F}_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} + \mathcal{F}_c\left\{\frac{\partial^2 u}{\partial y^2}\right\} = 0 \Rightarrow -\alpha^2 U(\alpha, y) - u_x(0, y) + \frac{\partial^2 U(\alpha,y)}{\partial y^2} = 0$$

1 $$\Rightarrow \frac{\partial^2 U(\alpha,y)}{\partial y^2} - \alpha^2 U(\alpha, y) = \sin y \qquad (*)$$

This is a nonhomogeneous ODE $\Rightarrow U = U_c + U_p$

First, we find the complementary solution U_c of the associated homogeneous equation

3 $$\frac{\partial^2 U(\alpha,y)}{\partial y^2} - \alpha^2 U(\alpha, y) = 0 \Rightarrow m^2 - \alpha^2 = 0 \Rightarrow m = \pm\alpha \Rightarrow U_c = c_1 \cosh \alpha y + c_2 \sinh \alpha y$$

Now, since the right side of $(*)$ is $\sin y$, then to find a particular solution U_p of $(*)$ we try with $U_p = A \sin y + B \cos y$ which we substitute in $(*)$ to get

$$-A \sin y - B \cos y - \alpha^2 (A \sin y + B \cos y) = \sin y$$

$$\Rightarrow -A(1 + \alpha^2) \sin y - B(1 + \alpha^2) \cos y = \sin y \Rightarrow -A(1 + \alpha^2) = 1 \ \text{ and } \ -B(1 + \alpha^2) = 0$$

3 $$\Rightarrow A = -\frac{1}{1+\alpha^2} \ \text{ and } \ B = 0 \Rightarrow U_p = -\frac{1}{1+\alpha^2} \sin y$$

2 Hence, $$U(\alpha, y) = U_c + U_p = c_1 \cosh \alpha y + c_2 \sinh \alpha y - \frac{1}{1+\alpha^2} \sin y \qquad (**)$$

Now, we take the Fourier cosine transform of the remaining boundary conditions

1 (a) $\mathcal{F}_c\{u(x,0)\} = \mathcal{F}_c\{0\} \Longrightarrow U(\alpha,0) = 0$

4 (b) $\mathcal{F}_c\{u(x,\pi)\} = \mathcal{F}_c\{f(x)\} \Longrightarrow U(\alpha,\pi) = \int_0^\infty f(x)\cos\alpha x\,dx = \int_0^1 \cos\alpha x\,dx = \frac{\sin\alpha}{\alpha}$

1 • Using (a) in $(**)$ gives: $c_1 = 0 \Longrightarrow U(\alpha,y) = c_2\sinh\alpha y - \frac{1}{1+\alpha^2}\sin y$

• Using (b) gives: $c_2\sinh\alpha\pi = \frac{\sin\alpha}{\alpha} \Longrightarrow c_2 = \frac{\sin\alpha}{\alpha\sinh\alpha\pi}$

2
$$\therefore U(\alpha,y) = \frac{\sin\alpha}{\alpha\sinh\alpha\pi}\sinh\alpha y - \frac{1}{1+\alpha^2}\sin y$$

$$\Longrightarrow u(x,y) = \mathcal{F}_c^{-1}\left\{\frac{\sin\alpha}{\alpha\sinh\alpha\pi}\sinh\alpha y - \frac{1}{1+\alpha^2}\sin y\right\}$$

4
$$= \frac{2}{\pi}\int_0^\infty \left[\frac{\sin\alpha}{\alpha\sinh\alpha\pi}\sinh\alpha y - \frac{1}{1+\alpha^2}\sin y\right]\cos\alpha x\,d\alpha$$