King Fahd University of Petroleum & Minerals
Department of Mathematics and Statistics
MATH 301 – Methods of Applied Mathematics
EXAM II
2014-2015 (141)

Wednesday, November 26, 2014 **Allowed Time: 100 min.**

ANSWER KEY	ID #:	
Instructor:	**Sec. #:**	**Serial #:**

Instructions:

1. Write neatly and legibly. You may lose points for messy work.

2. **Show all your work.** No points for answers without justification.

3. **Calculators and Mobiles are not allowed**.

4. Make sure that you have 6 different problems (6 pages + cover page).

Problem No.	Points	Maximum Pts
1		18
2		10
3		17
4		10
5		22
6		23
Total		100

Q1 (9+9 pts): Evaluate

2 (A) $\mathcal{L}\{e^{2t}\cos^2 t\} = \mathcal{L}\{\cos^2 t\}|_{s\to s-2}$

2 $$= \mathcal{L}\left\{\frac{1}{2}(1+\cos 2t)\right\}|_{s\to s-2}$$

4 $$= \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+4}\right)|_{s\to s-2}$$

1 $$= \frac{1}{2}\left(\frac{1}{s-2} + \frac{s-2}{(s-2)^2+4}\right)$$

2 (B) $\mathcal{L}^{-1}\left\{\frac{2s-3}{s^3+s^2+4s+4}\right\} = \mathcal{L}^{-1}\left\{\frac{2s-3}{(s+1)(s^2+4)}\right\}$

3 $$\underset{\text{partial fractions}}{=} \mathcal{L}^{-1}\left\{\frac{-1}{s+1} + \frac{s+1}{s^2+4}\right\}$$

1 $$= \mathcal{L}^{-1}\left\{\frac{-1}{s+1} + \frac{s}{s^2+4} + \frac{1}{s^2+4}\right\}$$

3 $$= -e^{-t} + \cos 2t + \frac{1}{2}\sin 2t$$

<u>Q2 (10 pts)</u>: Use the **<u>Convolution Theorem</u>** to find $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s-1)}\right\}$

<u>Solution</u>:

2 $$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s-1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\cdot\frac{1}{s-1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

3 $$= t * e^t$$

2 $$= \int_0^t \tau e^{t-\tau} d\tau$$

$$= \left[-\tau e^{t-\tau} - e^{t-\tau}\right]_{\tau=0}^{\tau=t}$$

$$= [-t-1] - [-e^t]$$

3 $$= e^t - t - 1$$

Q3 (17 pts): Use the Laplace transform to solve the IVP

$$y' + 2y = t\delta(t-3), \qquad y(0) = 1$$

Solution:

1 $$\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{t\delta(t-3)\}$$

5 $$\Rightarrow [sY(s) - y(0)] + 2Y(s) = -\frac{d}{ds}\mathcal{L}\{\delta(t-3)\}$$

$$\Rightarrow (s+2)Y(s) = 1 - \frac{d}{ds}(e^{-3s})$$

$$\Rightarrow (s+2)Y(s) = 1 + 3e^{-3s}$$

3 $$\Rightarrow Y(s) = \frac{1}{s+2} + \frac{3}{s+2}e^{-3s}$$

1 $$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{s+2}e^{-3s}\right\}$$

$$= e^{-2t} + 3e^{-2t}|_{t\to t-3}\mathcal{U}(t-3)$$

5 $$= e^{-2t} + 3e^{-2(t-3)}\mathcal{U}(t-3)$$

2 $$= \begin{cases} e^{-2t}, & 0 \le t < 3 \\ e^{-2t} + 3e^{-2(t-3)}, & t \ge 3 \end{cases}$$

Q4 (10 pts): Show that the functions $f_1(x) = 3x^4$ and $f_2(x) = \frac{1}{\sqrt{7}}(5-6x)$ are orthonormal on the interval $[0,1]$ (i.e. they are orthogonal and their norms equal 1).

Solution:

$$(f_1, f_2) = \int_0^1 f_1(x) f_2(x) dx = \int_0^1 [3x^4]\left[\frac{1}{\sqrt{7}}(5-6x)\right] dx$$

$$= \frac{3}{\sqrt{7}} \int_0^1 (5x^4 - 6x^5) dx$$

$$= \frac{3}{\sqrt{7}} [x^5 - x^6]_0^1 = 0$$

5 So, f_1 and f_2 are orthogonal.

$$\|f_1\|^2 = \int_0^1 f_1^2(x) dx = \int_0^1 [3x^4]^2 dx = 9\int_0^1 x^8 dx = [x^9]_0^1 = 1$$

$$\Rightarrow \|f_1\| = \sqrt{1} = 1$$

$$\|f_2\|^2 = \int_0^1 f_2^2(x) dx = \int_0^1 \left[\frac{1}{\sqrt{7}}(5-6x)\right]^2 dx$$

5 $$= \frac{1}{7}\int_0^1 (5-6x)^2 dx = -\frac{1}{126}[(5-6x)^3]_0^1 = 1$$

$$\Rightarrow \|f_2\| = \sqrt{1} = 1$$

Hence, f_1 and f_2 are orthonormal.

Q5 (14+8 pts): (A) Find the Fourier series of the function $f(x) = \begin{cases} 0, & -1 < x < 0 \\ x, & 0 \leq x < 1 \end{cases}$.

Solution:

3 $$\bullet\ a_0 = \frac{1}{1}\int_{-1}^{1} f(x)dx = \int_0^1 xdx = \left[\frac{1}{2}x^2\right]_0^1 = \frac{1}{2}$$

3 $$\bullet\ a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos n\pi x\, dx = \int_0^1 x\cos n\pi x\, dx = \left[\frac{x}{n\pi}\sin n\pi x + \frac{1}{n^2\pi^2}\cos n\pi x\right]_0^1 = \frac{(-1)^n - 1}{n^2\pi^2}$$

3 $$\bullet\ b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin n\pi x\, dx = \int_0^1 x\sin n\pi x\, dx = \left[-\frac{x}{n\pi}\cos n\pi x + \frac{1}{n^2\pi^2}\sin n\pi x\right]_0^1 = -\frac{(-1)^n}{n\pi}$$

Therefore, the Fourier series of f is

5 $$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos n\pi x + b_n \sin n\pi x) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2\pi^2}\cos n\pi x - \frac{(-1)^n}{n\pi}\sin n\pi x$$

(B) Find the sine series of the constant function $f(x) = 1, \quad 0 < x < 1$.

Solution:

3 $$b_n = \frac{2}{L}\int_0^1 f(x)\sin\frac{n\pi}{L}x\, dx = \frac{2}{1}\int_0^1 (1)\sin n\pi x\, dx$$

$$= 2\left[-\frac{1}{n\pi}\cos n\pi x\right]_0^1$$

2 $$= 2\left[\left(\frac{-(-1)^n}{n\pi}\right) - \left(\frac{-1}{n\pi}\right)\right] = \frac{2 - 2(-1)^n}{n\pi}$$

Therefore, the sine series of f is:

3 $$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x = \sum_{n=1}^{\infty} \frac{2 - 2(-1)^n}{n\pi}\sin n\pi x$$

<u>Q6 (17+6 pts)</u>: (A) Find the eigenvalues and the eigenfunctions of the Sturm-Liouville problem: $y'' + 4y' + 4\lambda y = 0, \quad y(0) = 0, y(1) = 0$

1 **<u>Solution</u>**: The auxiliary equation is: $m^2 + 4m + 4\lambda = 0 \Rightarrow m = -2 \pm 2\sqrt{1-\lambda}$

2 ◙ Case I: $1 - \lambda = 0 \Rightarrow m = -2, -2 \Rightarrow y(x) = c_1 e^{-2x} + c_2 x e^{-2x}$

1 • $y(0) = 0 \Rightarrow c_1 = 0 \Rightarrow y(x) = c_2 x e^{-2x}$

1 • $y(1) = 0 \Rightarrow c_2 e^{-2} = 0 \Rightarrow c_2 = 0 \Rightarrow y(x) \equiv 0$ the trivial solution

◙ Case II: $1 - \lambda > 0$, so we put $1 - \lambda = \alpha^2$ $(\alpha > 0) \Rightarrow m = -2 + 2\alpha, -2 - 2\alpha$

2 $$\Rightarrow y(x) = c_1 e^{(-2+2\alpha)x} + c_2 e^{(-2-2\alpha)x}$$

1 • $y(0) = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_2 = -c_1 \Rightarrow y(x) = c_1 e^{(-2+2\alpha)x} - c_1 e^{(-2-2\alpha)x}$

• $y(1) = 0 \Rightarrow c_1 e^{(-2+2\alpha)} - c_1 e^{(-2-2\alpha)} = 0 \Rightarrow c_1 \left(\underbrace{e^{(-2+2\alpha)} - e^{(-2-2\alpha)}}_{\neq 0} \right) = 0 \Rightarrow c_1 = 0$

1 $$\Rightarrow c_2 = 0 \Rightarrow y(x) \equiv 0 \quad \text{the trivial solution}$$

◙ Case III: $1 - \lambda < 0$, so we put $1 - \lambda = -\alpha^2$ $(\alpha > 0) \Rightarrow m = -2 \pm 2\alpha i$

2 $$\Rightarrow y(x) = e^{-2x}[c_1 \cos 2\alpha x + c_2 \sin 2\alpha x]$$

1 • $y(0) = 0 \Rightarrow c_1 = 0 \Rightarrow y(x) = c_2 e^{-2x} \sin 2\alpha x$

• $y(1) = 0 \Rightarrow c_2 e^{-2} \sin 2\alpha = 0$

Since we aim to get nontrivial solutions and $\sin 2\alpha$ may be zero, we consider $c_2 \neq 0$ and

1 $$\sin 2\alpha = 0 \Rightarrow 2\alpha = n\pi \Rightarrow \alpha = \frac{n\pi}{2}, \qquad n = 1,2,3, \dots$$

$$\Rightarrow \text{the nontrivial solutions are } y(x) = c_2 e^{-2x} \sin n\pi x.$$

2 So, as $1 - \lambda = -\alpha^2$, this BVP has eigenvalues: $\lambda_n = 1 + \left(\frac{n\pi}{2}\right)^2, \qquad n = 1,2,3, \dots$

2 and eigenfunctions: $y_n(x) = e^{-2x} \sin n\pi x, \qquad n = 1,2,3, \dots$

(B) Put the differential equation in self-adjoint form and then give an orthogonality relation.

<u>Solution</u>: Multiply the equation by $e^{\int 4dx} = e^{4x}$ to get: $e^{4x} y'' + 4e^{4x} y' + 4\lambda e^{4x} y = 0$

3 $$\Rightarrow \frac{d}{dx}[e^{4x} y'] + 4\lambda e^{4x} y = 0 \qquad \text{is now in self-adjoint form}$$

The BVP is a regular Sturm-Liouville Problem with $r(x) = e^{4x} / q(x) = 0 / p(x) = 4e^{4x}$

$\Rightarrow$ The orthogonality relation is: $\int_0^1 4e^{4x} [e^{-2x} \sin n\pi x][e^{-2x} \sin m\pi x] dx = 0$

3 $$\Rightarrow \int_0^1 \sin n\pi x \sin m\pi x \, dx = 0 \qquad n \neq m$$