1. **(10 points)** If $A < B$ and $y = 3e^{Ax} + 2e^{Bx}$ is a solution of the differential equation $y'' + 2y' - 3y = 0$. Find A and B.

$y' = 3Ae^{Ax} + 2Be^{Bx}$

$y'' = 3A^2e^{Ax} + 2B^2e^{Bx}$

$y'' + 2y' - 3y = (3A^2 + 6A - 9)e^{Ax} + (2B^2 + 4B - 6)e^{Bx} = 0$

So $3A^2 + 6A - 9 = 0$, $2B^2 + 4B - 6 = 0$

i.e. $(A-1)(A+3) = 0$, $(B-1)(B+3) = 0$.

Since $A < B$ we get $A = -3$ and $B = 1$

2. (**10 points**) Solve the initial value problem

$$\frac{dy}{dx} + xy = 2xe^{x^2/2},\ y(0) = 2.$$

DE is linear with

integrating factor $\rho = e^{\int x\,dx} = e^{x^2/2}$

Hence $\frac{d}{dx}(\rho y) = \rho\left(2xe^{x^2/2}\right)$

i.e. $\rho y = \int 2xe^{x^2}dx = e^{x^2} + C$

So $y = e^{-x^2/2}\left(e^{x^2} + C\right)$

$y(0) = 2$ gives $2 = 1 + C$ i.e. $C = 1$,

hence IVP has solution

$y = e^{-x^2/2}\left(e^{x^2} + 1\right)$

$\left(\text{or } y = e^{x^2/2} + e^{-x^2/2}\right)$

3. (**10 points**) Find the general solution of the differential equation

$$x\frac{dy}{dx} + 6y = 3x\,y^{4/3}.$$

DE is $\frac{dy}{dx} + \frac{6}{x}y = 3y^{4/3}$

Bernoulli with $r = 4/3$

Put $v = y^{1-4/3} = y^{-1/3}$, then

$\frac{dv}{dx} = -\frac{1}{3}y^{-4/3}\frac{dy}{dx}$

DE becomes

$\frac{dv}{dx} - \frac{2}{x}v = -1$, linear 1^{st} order

Integrating factor $\rho = e^{-\int \frac{2}{x}dx} = x^{-2}$

Hence $\frac{d}{dx}(\rho v) = -\rho$

i.e. $\rho v = -\int \rho\, dx$ and we get

$v = x^2\left(\frac{1}{x} + C\right)$

i.e. $y = \left(x + Cx^2\right)^{-3}$

4. **(10 points)** Use the method of Gauss-Jordan elemination to solve the system

$$x_1 + 3x_2 + 3x_3 = 13$$

$$2x_1 + 5x_2 + 4x_3 = 23$$

$$2x_1 + 7x_2 + 8x_3 = 29$$

Augmented matrix

$$\begin{bmatrix} 1 & 3 & 3 & 13 \\ 2 & 5 & 4 & 23 \\ 2 & 7 & 8 & 29 \end{bmatrix}.$$

$$\xrightarrow[R_3-2R_1]{R_2-2R_1} \begin{bmatrix} 1 & 3 & 3 & 13 \\ 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 \end{bmatrix} \xrightarrow{-R_2} \begin{bmatrix} 1 & 3 & 3 & 13 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$$\xrightarrow{R_3-R_2} \begin{bmatrix} 1 & 3 & 3 & 13 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1-3R_2} \underbrace{\begin{bmatrix} 1 & 0 & -3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{RREF}}$$

So $x_3 = t$, $x_2 = 3-2t$, $x_1 = 4+3t$

5. **(10 points)** Solve the initial value problem

$$\frac{dy}{dx} = \frac{xy + 3x - y - 3}{xy - 2x + 4y - 8}, \quad y(0) = 1$$

$$\frac{dy}{dx} = \frac{(x-1)(y+3)}{(x+4)(y-2)}$$ Separable

$$\frac{y-2}{y+3}\,dy = \frac{x-1}{x+4}\,dx$$

$$\frac{y+3-5}{y+3}\,dy = \frac{x+4-5}{x+4}\,dx$$

$$\int\left(1-\frac{5}{y+3}\right)dy = \int\left(1-\frac{5}{x+4}\right)dx$$

$$y - 5\ln|y+3| = x - 5\ln|x+4| + C$$

$y(0)=1$ gives $1 - 5\ln 4 = -5\ln 4 + C$ i.e. $C = 1$

So IVP has solution

$$y - 5\ln|y+3| = x - 5\ln|x+4| + 1$$

6. **(10 points)** Solve the initial value problem

$$\frac{dy}{dx} = \frac{x^2}{(1+x^3)^3}, \quad y(0) = 1.$$

Separable DE

$$dy = \frac{x^2\,dx}{(1+x^3)^3}$$

$$\int dy = \int \frac{x^2\,dx}{(1+x^3)^3}$$

Put $1+x^3 = v$ so $3x^2\,dx = dv$

$$y = \frac{1}{3}\int \frac{dv}{v^3} = -\frac{1}{6v^2} + C$$

i.e. $y = -\dfrac{1}{6(1+x^3)^2} + C$

$y(0) = 1$ gives $1 = -\frac{1}{6} + C$ i.e. $C = \frac{7}{6}$

IVP has solution

$$y = -\frac{1}{6(1+x^3)^2} + \frac{7}{6}$$

7. **(10 points)** Verify that the differential equation

$$(y^2 + \cos x - 1)\, dx + (2xy + \cos y + 1)\, dy = 0$$

is exact and then find its general solution.

Put $M = y^2 + \cos x - 1$, $N = 2xy + \cos y + 1$

then $M_y = 2y$ and $N_x = 2y$. Since $M_y = N_x$, DE is exact.

Hence there is a function F such that

$F_x = M$ and $F_y = N$.

We have $F_x = y^2 + \cos x - 1$

so $F = xy^2 + \sin x - x + g(y)$

$F_y = 2xy + g'(y) = 2xy + \cos y + 1$

So $g(y) = \int (\cos y + 1)\, dy = \sin y + y + C$

Solution is $F(x,y) = \text{constant}$ i.e.

$xy^2 + \sin x - x + \sin y + y + C = 0$

$\left(\text{or } xy^2 + \sin x + \sin y - x + y = C\right)$

8. **(10 points)** Determine for what values of k the system

$$x + 2y + z = 2$$

$$2x - y - 3z = 5$$

$$4x + 3y - z = k$$

has

a) a unique solution b) no solution c) infinitely many solution.

Augmented matrix

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & -1 & -3 & 5 \\ 4 & 3 & -1 & k \end{bmatrix}$$

$$\xrightarrow[R_3-4R_1]{R_2-2R_1} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -5 & -5 & 1 \\ 0 & -5 & -5 & k-8 \end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -5 & -5 & 1 \\ 0 & 0 & 0 & k-9 \end{bmatrix}$$

If $k-9=0$ (i.e. $k=9$), the system has infinitely many solutions because there will be one free variable (only 2 leading variables)

If $k-9 \neq 0$ (i.e. $k \neq 9$), the system is inconsistent.

There are no values of k for which the system has a unique solution.