King Fahd University of Petroleum and Minerals
Quiz 3 Math 202-131 Duration 25 minutes

Full Name:
ID:
Serial number:

Question 1. Use the existence and uniqueness theorem to find ~~the largest~~ an interval of definition I such that the IVP

$$\begin{cases} (\cot x)\, y'' + (\cos x)\, y & = 4x \\ y(2) = 0 \quad \text{and} \quad y'(2) & = 1 \end{cases}$$

has a unique solution.

$\cos x$, $4x$, $\cot x$ are continuous on $(0, \pi)$. But ~~on $(0,\pi)$~~ $\cot x$ must be $\neq 0$ $\forall\, x \in I$, so, we choose

$I = (0, \frac{\pi}{2})$ or $I = (\frac{\pi}{2}, \pi)$

but 2 must be in I, $\therefore$ $\boxed{I = (\frac{\pi}{2}, \pi)}$

(Note, $\cot(\frac{\pi}{2}) = 0$). and $I = (0, \frac{\pi}{2})$ is rejected

Question 2. Determine whether the functions $f_1(x) = e^x$, $f_2(x) = \sinh x$ and $f_3(x) = \cosh x$ are linearly independent or linearly dependent on $(-\infty, \infty)$.

$$W(f_1, f_2, f_3) = \begin{vmatrix} e^x & \sinh x & \cosh x \\ e^x & \cosh x & \sinh x \\ e^x & \sinh x & \cosh x \end{vmatrix} = e^x(\cosh^2 x - \sinh^2 x) - e^x(\sinh x \cosh x - \cosh x \sinh x) + e^x(\sinh^2 x - \cosh^2 x) = 0$$

$\therefore$ f_1, f_2 & f_3 are linearly independent

Question 3. Given that $y_1 = x$, $y_2 = x^2$ and $y_3 = x^2 \ln x$ are solutions of the homogeneous DE:

$$x^3y''' - 2x^2y'' + 4xy' - 4y = 0 \quad \text{on} \quad (0, \infty).$$

Find the general solution of this DE (justify your answer).

If y_1, y_2 & y_3 are linearly independent, then the general solution of this DE is:

$y = c_1 x + c_2 x^3 + c_3 x^2 \ln x$. Now, let us proof that y_1, y_2 & y_3 are linearly independent.

$$W(y_1, y_2, y_3) = \begin{vmatrix} x & x^2 & x^2 \ln x \\ 1 & 2x & 2x\ln x + x \\ 0 & 2 & 2\ln x + 3 \end{vmatrix} = \begin{aligned} & x(\cancel{4x\ln x} + 6x - \cancel{4x\ln x} - 2x) \\ & - (\cancel{2x^2\ln x} + 3x^2 - \cancel{2x^2 \ln x}) \\ & = 4x^2 - 3x^2 = x^2 \neq 0 \;\; \forall\, x \in (0, \infty) \end{aligned}$$

& so y_1, y_2, y_3 are linearly independent.

$$y'' + \frac{1}{x} y' - \frac{1}{x^2} y = 0$$

a

Question 4. The function $y_1 = x$ is a solution of $\boxed{x^2y'' + xy' - y = 0}$ on $(0, \infty)$. Use Reduction of Order method to find the general solution of this DE.

By reduction of order method, the general solution of the given DE is

$$y = c_1 y_1 \int y_1^{-2} e^{-\int a\,dx}\, dx$$

$$= c_1 x \int \frac{1}{x^2} e^{-\ln x}\, dx = c_1 x \int \frac{1}{x^3}\, dx = c_1 x \left[\frac{-1}{2x^2} + C\right]$$

$$= c_1 \frac{1}{x} + c_2 x$$

y_2 $\quad$ y_1

$\therefore$ $y = \frac{c_1}{x} + c_2 x$ is the general solution.