King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 202 - Final Exam - Term 141

Duration: 180 minutes

Name: KEY ID Number: ________

Section Number: ________ Serial Number: ________

Class Time: ________ Instructor's Name: ________

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write neatly and eligibly. You may lose points for messy work.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 12 pages of problems (Total of 12 Problems)

Page Number	Points	Maximum Points
1		12
2		14
3		12
4		12
5		10
6		14
7		14
8		10
9		9
10		11
11		10
12		12
Total		140

1. (**12 points**) Given $\lambda = 2 + i\sqrt{3}$ is an eigenvalue of the matrix $A = \begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix}$.

Find the general solution of the homogeneous linear system $X' = AX$.

$$\begin{bmatrix} 1-i\sqrt{3} & 2 \\ -2 & -1-i\sqrt{3} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{2pts} \Rightarrow$$

$$(1-i\sqrt{3})k_1 + 2k_2 = 0 \qquad \Rightarrow k_2 = -\frac{1}{2}(1-i\sqrt{3})k_1$$

$$-2k_1 + (-1-i\sqrt{3})k_2 = 0$$

2pts

Let $k_1 = -2 \Rightarrow k_2 = 1-i\sqrt{3}$

$$\Rightarrow B = \begin{bmatrix} -2 \\ 1-i\sqrt{3} \end{bmatrix} = \underbrace{\begin{bmatrix} -2 \\ 1 \end{bmatrix}}_{B_1} + i\underbrace{\begin{bmatrix} 0 \\ -\sqrt{3} \end{bmatrix}}_{B_2} \quad \text{2pts}$$

$$X_1 = [B_1 \cos\beta t - B_2 \sin\beta t]e^{\alpha t}$$

$$= \left[\begin{pmatrix} -2 \\ 1 \end{pmatrix}\cos\sqrt{3}t - \begin{pmatrix} 0 \\ -\sqrt{3} \end{pmatrix}\sin\sqrt{3}t\right]e^{2t} \quad \text{2pts}$$

$$X_2 = [B_2 \cos\beta t + B_1 \sin\beta t]e^{\alpha t}$$

$$= \left[\begin{pmatrix} 0 \\ -\sqrt{3} \end{pmatrix}\cos\sqrt{3}t + \begin{pmatrix} -2 \\ 1 \end{pmatrix}\sin\sqrt{3}t\right]e^{2t} \quad \text{2pts}$$

2pts

General Solution $X = c_1X_1 + c_2X_2$

2. (**14 points**) Given $X_c = c_1\begin{pmatrix}1\\1\end{pmatrix}e^t + c_2\begin{pmatrix}1\\3\end{pmatrix}e^{-t}$ is the complementary function of the nonhomogeneous system $X' = AX + \begin{pmatrix}0\\4\end{pmatrix}t$. Use variation of parameters to find a particular solution X_p.

$X_1 = \begin{pmatrix}1\\1\end{pmatrix}e^t$ and $X_2 = \begin{pmatrix}1\\3\end{pmatrix}e^{-t}$ 2 pts

$\Phi(t) = \begin{pmatrix}e^t & e^{-t}\\ e^t & 3e^{-t}\end{pmatrix}$ 1 pt $\Rightarrow |\Phi(t)| = 3 - 1 = 2 \neq 0$ 1 pt

$\Phi^{-1}(t) = \frac{1}{2}\begin{pmatrix}3e^{-t} & -e^{-t}\\ -e^t & e^t\end{pmatrix} = \begin{pmatrix}\frac{3}{2}e^{-t} & -\frac{1}{2}e^{-t}\\ -\frac{1}{2}e^t & \frac{1}{2}e^t\end{pmatrix}$ 2 pts

$\Phi^{-1}(t)F(t) = \begin{pmatrix}\frac{3}{2}e^{-t} & -\frac{1}{2}e^{-t}\\ -\frac{1}{2}e^t & \frac{1}{2}e^t\end{pmatrix}\begin{pmatrix}0\\4\end{pmatrix}t = \begin{pmatrix}-2te^{-t}\\ 2te^t\end{pmatrix}$ 2 pts

We need to integrate $-2te^{-t}$ & $2te^t$ to get

$\int -2te^{-t}\,dt = 2te^{-t} + 2e^{-t}$ 1 pt

$\int 2te^t\,dt = 2te^t - 2e^t$ 1 pt

$\Rightarrow \int \Phi^{-1}(t)F(t)\,dt = \begin{pmatrix}2te^{-t} + 2e^{-t}\\ 2te^t - 2e^t\end{pmatrix}$ 1 pt

$\Rightarrow X_p = \Phi(t)\int \Phi^{-1}(t)F(t)\,dt = \begin{pmatrix}e^t & e^{-t}\\ e^t & 3e^{-t}\end{pmatrix}\begin{pmatrix}2te^{-t} + 2e^{-t}\\ 2te^t - 2e^t\end{pmatrix}$ 2 pts

$= \begin{pmatrix}2t + 2 + 2t - 2\\ 2t + 2 + 6t - 6\end{pmatrix} = \begin{pmatrix}4t\\ 8t - 4\end{pmatrix}$ 1 pt

3. (**12 points**) Use matrix exponential to solve the initial-value problem
$X' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} X$ subject to $X(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \Rightarrow A^3 = \begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix} \Rightarrow \cdots \Rightarrow$$

$$A^n = \begin{pmatrix} 1 & 0 \\ 0 & 2^n \end{pmatrix}$$ 3pts

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots + \frac{1}{n!}A^nt^n + \cdots$$ 1pt

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} t + \frac{1}{2!}\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} t^2 + \frac{1}{3!}\begin{pmatrix} 1 & 0 \\ 0 & 2^3 \end{pmatrix} t^3 + \cdots$$

$$+ \frac{1}{n!}\begin{pmatrix} 1 & 0 \\ 0 & 2^n \end{pmatrix} t^n + \cdots$$

$$= \begin{pmatrix} 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \cdots + \frac{1}{n!}t^n + \cdots & 0 \\ 0 & 1 + (2t) + \frac{1}{2!}(2t)^2 + \frac{1}{3!}(2t)^3 + \cdots + \frac{1}{n!}(2t)^n + \cdots \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{n=0}^{\infty} \frac{t^n}{n!} & 0 \\ 0 & \sum_{n=0}^{\infty} \frac{(2t)^n}{n!} \end{pmatrix} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}$$ 4pts

Soln $X = e^{At} C_0 = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 e^t \\ c_2 e^{2t} \end{pmatrix}$ 2pts

$X(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \Rightarrow c_1 = 1$ and $c_2 = 0 \Rightarrow X = \begin{pmatrix} e^t \\ 0 \end{pmatrix}$ 2pts

4. Given a Caucy-Euler equation

$$x^2y'' - 4xy' + 6y = \ln x^2 \qquad (I)$$

a) (**4 points**)Use a suitable substitution to transform equation (I) into an equation with constant coefficients.

Let $x = e^t \Rightarrow t = \ln x \Rightarrow \frac{dy}{dx} = \frac{1}{x}\frac{dy}{dt}$, $\frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$ (1pt) (2pts)

$\Rightarrow x^2y'' - 4xy' + 6y = \ln x^2$ can be transformed into

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 4\frac{dy}{dt} + 6y = 2t$$

$$\Rightarrow \frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = 2t \qquad (1pt)$$

b) (**8 points**)Use the new equation obtained in (a) to find the general solution of equation (I).

$m^2 - 5m + 6 = 0 \Rightarrow m = 2, 3 \Rightarrow y_c = c_1e^{2t} + c_2e^{3t}$ (2pts)

for y_p; $y_p = A + Bt$ (2pts)

$y_p' = B \Rightarrow \frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = -5B + 6A + 6Bt = 2t$

$y_p'' = 0$

$\Rightarrow 6B = 2$ and $-5B + 6A = 0 \Rightarrow B = \frac{1}{3}$ and $A = \frac{5}{18}$ (2pts)

$\therefore y_p = \frac{5}{18} + \frac{1}{3}t$ -- (1pt)

$\therefore y = y_c + y_p = c_1e^{2t} + c_2e^{3t} + \frac{5}{18} + \frac{1}{3}t$

$= c_1x^2 + c_2x^3 + \frac{5}{18} + \frac{1}{3}\ln x$ -- (1pt)

5. **(10 points)** If $X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$ is a solution of the system $X' = \begin{pmatrix} -1 & 3 \\ -3 & 5 \end{pmatrix} X$ which corresponds to the eigenvalue $\lambda = 2$ of multiplicity 2. Find a second linearly independent solution.

$K_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\lambda = 2$ 2pts

Soln $X_2 = K_1 t e^{2t} + P e^{2t}$ 2pts

For P, we need to solve $\begin{pmatrix} -3 & 3 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ 2pts

$\Rightarrow \quad -3p_1 + 3p_2 = 1$

let $p_2 = 0 \Rightarrow p_1 = -\frac{1}{3} \Rightarrow P = \begin{pmatrix} -\frac{1}{3} \\ 0 \end{pmatrix}$ 2pts

$\Rightarrow X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{2t} + \begin{pmatrix} -\frac{1}{3} \\ 0 \end{pmatrix} e^{2t}$ 2pts

6. (**14 points**) Explain why $x = 0$ is an ordinary point of the equation $y'' - xy' - y = 0$. Then find two linearly independent power series solutions of the given equation about $x = 0$.

$x=0$ is an ordinary point since the coefficients (1pt)

$a_2(0)=1 \neq 0$, $a_1(x) = -x$ and $a_0(x) = -1$ are analytic at $x=0$

Soln $y = \sum_{n=0}^{\infty} c_n x^n \Rightarrow y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \Rightarrow y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$ (2pts)

$$\Rightarrow y'' - xy' - y = \underbrace{\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}}_{k=n-2} - \underbrace{\sum_{n=1}^{\infty} n c_n x^n}_{k=n} - \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n}$$

$$= \sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k - \sum_{k=1}^{\infty} k c_k x^k - \sum_{k=0}^{\infty} c_k x^k$$ (3pts)

$$= (2c_2 - c_0) + \sum_{k=1}^{\infty} [(k+2)(k+1) c_{k+2} - (k+1) c_k] x^k = 0$$ (2pts)

$\Rightarrow c_2 = \frac{1}{2} c_0$ and $c_{k+2} = \frac{1}{k+2} c_k$ $\quad k = 1, 2, 3, \ldots$

$k=1 \Rightarrow c_3 = \frac{1}{3} c_1$ $\qquad k=2 \Rightarrow c_4 = \frac{1}{4} c_2 = \frac{1}{2 \cdot 4} c_0$

$k=3 \Rightarrow c_5 = \frac{1}{5} c_3 = \frac{1}{3 \cdot 5} c_1$ $\qquad k=4 \Rightarrow c_6 = \frac{1}{6} c_4 = \frac{1}{2 \cdot 4 \cdot 6} c_0$

$k=5 \Rightarrow c_7 = \frac{1}{7} c_5 = \frac{1}{3 \cdot 5 \cdot 7} c_1$ $\qquad k=6 \Rightarrow c_8 = \frac{1}{8} c_6 = \frac{1}{2 \cdot 4 \cdot 6 \cdot 8} c_0$ (2pts)

$\vdots \qquad \vdots$

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots$$

$$= c_0 + c_1 x + \frac{1}{2} c_0 x^2 + \frac{1}{3} c_1 x^3 + \frac{1}{2 \cdot 4} c_0 x^4 + \frac{1}{3 \cdot 5} c_1 x^5 + \frac{1}{2 \cdot 4 \cdot 6} c_0 x^6 + \cdots$$

$$= c_0 \left(1 + \frac{1}{2} x^2 + \frac{1}{2 \cdot 4} x^4 + \frac{1}{2 \cdot 4 \cdot 6} x^6 + \cdots\right) + c_1 \left(x + \frac{1}{3} x^3 + \frac{1}{3 \cdot 5} x^5 + \frac{1}{3 \cdot 5 \cdot 7} x^7 + \cdots\right)$$ (3 pts)

$$\Rightarrow y_1 = 1 + \frac{1}{2} x^2 + \frac{1}{2 \cdot 4} x^4 + \frac{1}{2 \cdot 4 \cdot 6} x^6 + \cdots = 1 + \sum_{n=1}^{\infty} \frac{1}{2^n (n!)} x^{2n}$$

$$\& \ y_2 = x + \frac{1}{3} x^3 + \frac{1}{3 \cdot 5} x^5 + \frac{1}{3 \cdot 5 \cdot 7} x^7 + \cdots = x + \sum_{n=1}^{\infty} \frac{1}{3 \cdot 5 \cdot 7 \cdots (2n+1)} x^{2n+1}$$

(1pt)

Soln $y = C_1 y_1 + C_2 y_2$

7. (**14 points**) Find the first four terms of the series solution of the equation $2x^2y'' - 3xy' + (3+x)y = 0$ which corresponds to the smaller indicial root of the singularity $x = 0$.

$$y = \sum_{n=0}^{\infty} c_n x^{n+r} \Rightarrow y' = \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1} \Rightarrow y'' = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2}$$ 2pts

$$\Rightarrow 2x^2y'' - 3xy' + (3+x)y = \sum_{n=0}^{\infty} 2c_n(n+r)(n+r-1)x^{n+r} - \sum_{n=0}^{\infty} 3c_n(n+r)x^{n+r} + \sum_{n=0}^{\infty} 3c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+1}$$ 2pts

$$= x^r \Big[\underbrace{\sum_{n=0}^{\infty} (2c_n(n+r)(n+r-1) - 3c_n(n+r) + 3c_n) x^n}_{k=n} + \underbrace{\sum_{n=0}^{\infty} c_n x^{n+1}}_{k=n+1} \Big]$$ 2pts

$$= x^r \Big[\sum_{k=0}^{\infty} (2c_k(k+r)(k+r-1) - 3c_k(k+r) + 3c_k) x^k + \sum_{k=1}^{\infty} c_{k-1} x^k \Big]$$

$$= x^r \Big[(2c_0(r)(r-1) - 3c_0(r) + 3c_0) + \sum_{k=1}^{\infty} \big[2c_k(k+r)(k+r-1) - 3c_k(k+r) + 3c_k + c_{k-1} \big] x^k \Big] = 0$$ 2pts

$$\Rightarrow (2r(r-1) - 3r + 3)c_0 = 0 \Rightarrow (2r-3)(r-1) = 0 \Rightarrow r = 3/2 \text{ and } \underline{r=1} \text{ (smaller)} \text{ \& } c_0 \text{ is arbitrary}$$

recurrence relation $2c_k(k+1)k - 3c_k(k+1) + 3c_k + c_{k-1} = 0 \quad k = 1, 2, 3, \dots$ 2pts

$$\Rightarrow c_k = \frac{-1}{k(2k-1)} c_{k-1}, \quad k = 1, 2, 3, \dots$$

$$k=1 \Rightarrow c_1 = \frac{-1}{1(1)} c_0 = -c_0 \qquad k=2 \Rightarrow c_2 = \frac{-1}{2(3)} c_1 = \frac{1}{6} c_0$$

$$c_3 = \frac{-1}{3(5)} c_2 = \frac{-1}{90} c_0 \quad \dots$$

Soln $$y = \sum_{n=0}^{\infty} c_n x^{n+1} = c_0 x + c_1 x^2 + c_2 x^3 + c_3 x^4 + \dots$$

$$= c_0 x - c_0 x^2 + \frac{1}{6} c_0 x^3 - \frac{1}{90} c_0 x^4 + \dots$$ 4pts

$$= c_0 \underbrace{\left(x - x^2 + \frac{1}{6}x^3 - \frac{1}{90}x^4 + \dots \right)}_{y_1}$$

8. a) (**5 points**) Determine a linear homogeneous differential equation with the least order and with constant coefficients and having the general solution

$$y = c_1 + c_2e^{-x} + c_3x\,e^{-x} + c_4e^{2x}\cos 3x + c_5e^{2x}\sin 3x$$

1pt $\to D$ $(D+1)^2$ 1pt $(D^2-4D+13)$ 2pts

Equation $D(D+1)^2(D^2-4D+13)\,y = 0$ 1pt

b) (**5 points**) Classify each singular point of the equation

$$x(x-1)^3\,y'' + (x-1)^2y' + 6xy = 0$$

as regular or irregular.

Singular points: $x(x-1)^3 = 0 \Rightarrow x = 0, 1$ 1pt

$P(x) = \dfrac{1}{x(x-1)}$ & $Q(x) = \dfrac{6}{(x-1)^3}$ 2pts

$\Rightarrow x = 0$ is a regular singular point 1pt

$x = 1$ is an irregular singular point 1pt

9. a) (**5 points**)Find an integrating factor to make the differential equation

$$y(10x + 3y + 5)\,dx + (5x + 3y)\,dy = 0$$

exact. (**Do not solve the equation**)

$M(x,y) = y(10x+3y+5) \Rightarrow M_y = 10x+6y+5$

$N(x,y) = 5x+3y \Rightarrow N_x = 5$ 2pts

$\dfrac{M_y - N_x}{N} = \dfrac{10x+6y}{5x+3y} = 2$ (function of x only) 1pt

$\Rightarrow \mu(x) = e^{\int \frac{M_y - N_x}{N} dx} = e^{\int 2dx} = e^{2x}$ 2pts

b) (**4 points**) Given that the vectors $X_1 = \begin{pmatrix} 6 \\ -1 \\ -5 \end{pmatrix} e^{-t}$, $X_2 = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} e^{-2t}$,

and $X_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} e^{3t}$ are the solutions of a system $X' = AX$.

Determine whether these vectors form a fundamental set of solutions on the interval $(-\infty, \infty)$.

We need to check that X_1, X_2 and X_3 are LI. 1pt

$W(X_1, X_2, X_3) = e^{-t} e^{-2t} e^{3t} \begin{vmatrix} 6 & -3 & 2 \\ -1 & 1 & 1 \\ -5 & 1 & 1 \end{vmatrix}$

$= 6\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} -3 & 2 \\ 1 & 1 \end{vmatrix} - 5\begin{vmatrix} -3 & 2 \\ 1 & 1 \end{vmatrix} = -4(-5) = 20 \neq 0$ 2pts

$\therefore X_1, X_2$ and X_3 are LI

$\therefore X_1, X_2$ and X_3 form a fundamental set of solutions 1pt

10. (**11 points**) Show that the equation $(1+y^2)\,dx - (2x^3y + xy)\,dy = 0$ is of Bernoulli's type. Then find its general solution.

$$\Rightarrow (1+y^2)\frac{dx}{dy} - yx = 2yx^3$$

$$\Rightarrow \frac{dx}{dy} - \frac{y}{1+y^2}x = \frac{2y}{1+y^2}x^3 \quad \text{Bernoulli} \quad 2\text{pts}$$

$$n=3 \Rightarrow u = x^{-2} \Rightarrow x = u^{-1/2} \Rightarrow \frac{dx}{dy} = \frac{-1}{2}u^{-3/2}\frac{du}{dy} \quad 2\text{pts}$$

New equation $\quad \frac{-1}{2}u^{-3/2}\frac{du}{dy} - \frac{y}{1+y^2}u^{-1/2} = \frac{2y}{1+y^2}u^{-3/2} \quad * (-2u^{3/2})$

$$\Rightarrow \frac{du}{dy} + \frac{2y}{1+y^2}u = \frac{-4y}{1+y^2} \quad \text{linear in } u \quad 2\text{pts}$$

$$\text{I.f.} = e^{\int \frac{2y}{1+y^2}dy} = 1+y^2 \quad 1\text{pt}$$

$$\Rightarrow (1+y^2)\frac{du}{dy} + 2yu = -4y \quad 2\text{pts}$$

$$\Rightarrow \frac{d}{dy}\left[(1+y^2)u\right] = -4y$$

$$\Rightarrow (1+y^2)u = \int -4y\,dy = -2y^2 + C$$

$$\Rightarrow u = \frac{-2y^2}{1+y^2} + \frac{C}{1+y^2}$$

$$\Rightarrow x^{-2} = \frac{-2y^2 + C}{1+y^2} \quad 2\text{pts}$$

11. (**10 points**) Given that $y = c_1 x + c_2 u(x) x$ is the general solution of the equation

$$x^2 y'' - x(x+2)y' + (x+2)y = 0, \quad x > 0$$

Find $u(x)$.

$$u(x) = \int \frac{e^{-\int p(x)\,dx}}{y_1^2(x)}\,dx \quad \text{where} \quad y_1(x) = x$$ (2pts) (2pts)

$$p(x) = \frac{-x(x+2)}{x^2} = -\left(1 + \frac{2}{x}\right)$$ (2pts)

$$\Rightarrow e^{-\int p(x)\,dx} = e^{\int (1+\frac{2}{x})\,dx} = x^2 e^x$$ (2pts)

$$\Rightarrow u(x) = \int \frac{e^{-\int p(x)\,dx}}{y_1^2(x)}\,dx = \int \frac{e^x \cdot x^2}{x^2}\,dx = e^x$$ (2pts)

12. (**12 points**) Solve $y'' - 4y' + 4y = (x+1)e^{2x}$ by using variation of parameters. (other methods of solutions will not be accepted).

First we solve $y'' - 4y' + 4y = 0$ to get

$m^2 - 4m + 4 = 0 \Rightarrow m = 2, 2 \Rightarrow y_c = c_1 e^{2x} + c_2 x e^{2x}$ 2pts

$$W(x_1, x_2) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} = e^{4x} \neq 0$$ 2pts

$$\Rightarrow u_1' = \frac{W_1}{W} = \frac{\begin{vmatrix} 0 & xe^{2x} \\ (x+1)e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix}}{e^{4x}} = \frac{-x(x+1)e^{4x}}{e^{4x}}$$

$$= -x(x+1) \Rightarrow u_1 = \frac{-x^3}{3} - \frac{x^2}{2}$$ 3pts

$$u_2' = \frac{W_2}{W} = \frac{\begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (x+1)e^{2x} \end{vmatrix}}{e^{4x}} = \frac{(x+1)e^{4x}}{e^{4x}} = (x+1)$$

$$\Rightarrow u_2 = \frac{x^2}{2} + x$$ 3pts

$$\Rightarrow y_p = u_1 y_1 + u_2 y_2 = \left(\frac{-x^3}{3} - \frac{x^2}{2}\right)e^{2x} + \left(\frac{x^2}{2} + x\right)xe^{2x}$$

$$= \left(-\frac{1}{3} + \frac{1}{2}\right)x^3 e^{2x} + \left(1 - \frac{1}{2}\right)x^2 e^{2x} = \frac{1}{6}x^3 e^{2x} + \frac{1}{2}x^2 e^{2x}$$ 1pt

$$\therefore y = y_c + y_p = c_1 e^{2x} + c_2 x e^{2x} + \frac{1}{2}x^2 e^{2x} + \frac{1}{6}x^3 e^{2x}$$ 1pt