King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 202 - Exam II - Term 141

Duration: 120 minutes

Name:____________________ ID Number:____________________

Section Number:_______________ Serial Number:________________

Class Time: __________________ Instructor's Name:______________

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write neatly and eligibly. You may lose points for messy work.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 9 pages of problems (Total of 9 Problems)

Page Number	Points	Maximum Points
1		8
2		11
3		9
4		11
5		9
6		11
7		16
8		8
9		17
Total		100

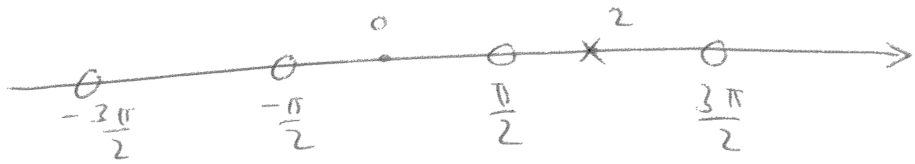
1. a)(**4 points**) Find an interval centered about $x = 0$ for which the initial-value problem

$$(x-2)y'' + (\tan x)y = x,\ \ y(0) = 1, y'(0) = 0$$

has a unique solution.

3 pts

$a_2(x) = x-2$, $a_1(x) = 0$, and $g(x) = x$ are continuous on $(-\infty, \infty)$, while $a_2(x) = \tan x$ is continuous for all $x \neq (2n+1)\frac{\pi}{2}$. Also, $a_2(x) = x-2 = 0$ at $x = 2$.



1pt $\Rightarrow$ The required interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

b) (**4 points**) Given that $y = c_1x^2 + c_2x^4 + 3$ is a two-parameter family of solutions of a differential equation on the interval $(-\infty, \infty)$. Determine whether a member of the family can be found that satisfies the boundary conditions $y(-1) = 0, y(1) = 4$.

$y(-1) = 0$ gives $c_1 + c_2 + 3 = 0$ — 1pt

$y(1) = 4$ gives $c_1 + c_2 + 3 = 4$ — 1pt

subtract $0 = 4$ — 1pt

which is not possible, so there is no member of the family can be found — 1pt

2. a) (**5 points**) Determine whether the set of functions

$$f_1(x) = 2\sqrt{x} + 3,\ f_2(x) = \sqrt{x} + 2x,\ f_3(x) = 4x - 3,\ f_4(x) = x^2 + 1$$

is linearly independent on the interval $(0, \infty)$.

observe that

$$2\sqrt{x} + 3 = 2(\sqrt{x} + 2x) - (4x - 3) + 0 \cdot (x^2 + 1)$$

i.e $f_1(x) = 2 \cdot f_2(x) + (-1) f_3(x) + 0 \cdot f_4(x)$ 4 pts

for every x in the interval $(0, \infty)$

Therefore, the given set is linearly dependent on $(0, \infty)$. 1pt

b) (**6 points**) Let L be a linear differential operator such that y_{p_1} and y_{p_2} are respectively, particular solutions of $L(y) = -\frac{1}{3}\cos^2 x$ and $L(y) = 4\sin^2 x$. Find a particular solution of $L(y) = \cos 2x$.

Since L is linear, then

$L(y_{p_1}) = -\frac{1}{3}\cos^2 x \Rightarrow L(-3y_{p_1}) = \cos^2 x$ 1pt

$L(y_{p_2}) = 4\sin^2 x \Rightarrow L(-\frac{1}{4}y_{p_2}) = -\sin^2 x$ 1pt

and so $L(-3y_{p_1} - \frac{1}{4}y_{p_2}) = L(-3y_{p_1}) + L(-4y_{p_2})$

$= \cos^2 x - \sin^2 x = \cos 2x$ 2pts

Therefore $y_p = -3y_{p_1} - \frac{1}{4}y_{p_2}$ is a particular solution of $L(y) = \cos 2x$. 2pts

3. (**9 points**) Without solving the differential equation, verify that $y = c_1x^{-1} + c_2x - \ln x$ is the general solution of $x^2y'' + xy' - y = \ln x,\ x > 0$.

step 1 We verify that the functions $y_1 = x^{-1}$ and $y_2 = x$ form a fundamental set of solutions of the associated homogeneous equation $x^2y'' + xy' - y = 0$ 2pts

$y_1 = x^{-1} \Rightarrow y_1' = -x^{-2} \Rightarrow y_1'' = 2x^{-3}$

$\Rightarrow x^2(2x^{-3}) + x(-x^{-2}) - x^{-1} = 2x^{-1} - x^{-1} - x^{-1} = 0.$ 1pt

$y_2 = x \Rightarrow y_2' = 1 \Rightarrow y_2'' = 0 \Rightarrow x^2(0) + x(1) - x = 0$ 1pt

Also, $W(x^{-1}, x) = \begin{vmatrix} x^{-1} & x \\ -x^{-2} & 1 \end{vmatrix} = 2x^{-1} \neq 0, \quad x > 0.$ 2pts

1pt

step 2 We verify that $y_p = -\ln x$ is a particular solution.

$y_p = -\ln x \Rightarrow y_p' = -\frac{1}{x} \Rightarrow y_p'' = \frac{1}{x^2}$

$\Rightarrow x^2\left(\frac{1}{x^2}\right) + x\left(-\frac{1}{x}\right) - \ln x = -\ln x$ -- 2pts

4. (**11 points**) Given that $y_1 = x - 1$ is a solution of the differential equation $(2-2x+x^2)y'' - 2(x-1)y' + 2y = 0$ on $(-\infty, \infty)$, find a second solution y_2 that is linearly independent of y_1.

From the standard form of the equation,

$$y'' - \frac{2(x-1)}{2-2x+x^2}\, y' + \frac{2}{2-2x+x^2}\, y = 0,$$ 2 pts

we have $P(x) = -\dfrac{2x-2}{x^2-2x+2}$ — 1 pt and $Q(x) = \dfrac{2}{x^2-2x+2}$

are continuous on $(-\infty, \infty)$.

$\Rightarrow$ A second linearly independent solution is

$$y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\, dx$$ — 2 pts

$$= (x-1)\int \frac{e^{\int \frac{2x-2}{x^2-2x+2}}}{(x-1)^2}\, dx$$ — 1 pt

$$= (x-1)\int \frac{e^{\ln(x^2-2x+2)}}{(x-1)^2}\, dx$$ — 1 pt

1 pt

$$= (x-1)\int \frac{x^2-2x+2}{(x-1)^2}\, dx = (x-1)\int \frac{(x-1)^2+1}{(x-1)^2}\, dx$$

$$= (x-1)\int \left(1 + \frac{1}{(x-1)^2}\right) dx = (x-1)\left[x - \frac{1}{x-1}\right]$$

$$= x^2 - x - 1$$ -- 3 pts

5. (**9 points**) Solve $D^3(D-1)(D+2)^2(D^2+D+1)^2y = 0$

The auxiliary eqn is

$$m^3(m-1)(m+2)^2(m^2+m+1) = 0 \quad \text{— 2 pts}$$

$$\Rightarrow m = 0, 0, 0, 1, -2, -2, -\tfrac{1}{2} \pm i\tfrac{\sqrt{3}}{2}, -\tfrac{1}{2} \pm i\tfrac{\sqrt{3}}{2} \quad \text{— 2 pts}$$

$\Rightarrow$ The general solution is

$$y = c_1 + c_2 x + c_3 x^2 + c_4 e^{x} + c_5 e^{-2x} + c_6 x e^{-2x} + c_7 e^{-\frac{1}{2}x}\cos\left(\tfrac{\sqrt{3}}{2}x\right) + c_8 e^{-\frac{1}{2}x}\sin\left(\tfrac{\sqrt{3}}{2}x\right) + c_9 x e^{-\frac{1}{2}x}\cos\left(\tfrac{\sqrt{3}}{2}x\right) + c_{10} x e^{-\frac{1}{2}x}\sin\left(\tfrac{\sqrt{3}}{2}x\right)$$

5 pts

6. (**11 points**) Determine the form of a particular solution for
$(D^2 - 2D + 2)y = x^2e^x \sin x - 2xe^x \cos x$. **(Do not evaluate the constants)**

Step① we find y_c

Auxiliary equation $m^2-2m+2=0 \Rightarrow$

$m = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i \Rightarrow y_c = c_1 e^x \cos x + c_2 e^x \sin x$ (A) — 2 pts

Step② A differential operator $(D^2-2D+2)^3$ annihilates the RHS $x^2e^x \sin x - 2xe^x \cos x \Rightarrow$

$(D^2-2D+2)^3(D^2-2D+2)y = 0$ (B)

3 pts

$\Rightarrow (D^2-2D+2)^4 y = 0$

$\Rightarrow$ The roots of the auxiliary eqn. $(m^2-2m+2)^4 = 0$ of (B) are $1 \pm i$ of multiplicity 4 — 1 pt

Thus $y = \boxed{c_1 e^x \cos x + c_2 e^x \sin x} + c_3 x e^x \cos x + c_4 x e^x \sin x$
$+ c_5 x^2 e^x \cos x + c_6 x^2 e^x \sin x + c_7 x^3 e^x \cos x + c_8 x^3 e^x \sin x$ — 3 pts (C)

From (A) and (C) we arrive at the form of y_p:

$y_p = A x e^x \cos x + B x e^x \sin x + C x^2 e^x \cos x + D x^2 e^x \sin x$
$+ E x^3 e^x \cos x + F x^3 e^x \sin x$ — 2 pts

7. (**16 points**) Solve $y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$, $x > 0$

We use the method of variation of Parameters

The auxiliary equation $m^2 - 6m + 9 = (m-3)^2 = 0$

yields $m_1 = m_2 = 3 \Rightarrow y_c = c_1 e^{3x} + c_2 x e^{3x}$ — 2 pts

The given DE is already in standard form,

using $y_1 = e^{3x}$, $y_2 = x e^{3x}$, and $f(x) = \frac{e^{3x}}{x^2}$,

let $y_p = u_1(x)\,(e^{3x}) + u_2(x)\,(x e^{3x})$ — 2 pts

$$W(y_1, y_2) = \begin{vmatrix} e^{3x} & x e^{3x} \\ 3e^{3x} & 3xe^{3x} + e^{3x} \end{vmatrix} = e^{6x},$$ — 2 pts

$$W_1 = \begin{vmatrix} 0 & x e^{3x} \\ \frac{e^{3x}}{x^2} & 3xe^{3x} + e^{3x} \end{vmatrix} = -\frac{e^{6x}}{x}$$ — 2 pts

$$W_2 = \begin{vmatrix} e^{3x} & 0 \\ 3e^{3x} & \frac{e^{3x}}{x^2} \end{vmatrix} = \frac{e^{6x}}{x^2}$$ — 2 pts

$u_1' = \frac{W_1}{W} = -\frac{1}{x} \Rightarrow u_1 = -\ln x$ — 2 pts

$u_2' = \frac{W_2}{W} = \frac{1}{x^2} \Rightarrow u_2 = \frac{-1}{x}$ — 2 pts

$\Rightarrow y_p = (-\ln x)(e^{3x}) + (-\frac{1}{x})(x e^{3x}) = -e^{3x}\ln x \boxed{-e^{3x}}$ in y_c — 2 pts

$\Rightarrow y_p = -e^{3x}\ln x$

8. (**8 points**) Solve the boundary-value problem $x^2y'' + 3xy' + 4y = 0$ subject to $y(1) = 4$ and $y(e^{\pi/2\sqrt{3}}) = 0$ on the interval $(0, \infty)$.

This is a Cauchy-Euler eqn.

let $y = x^m \Rightarrow y' = m x^{m-1}$, $y'' = m(m-1)x^{m-2}$

and substitute in the given DE, we obtain

$m(m-1)x^m + 3mx^m + 4x^m = x^m[m^2 + 2m + 4] = 0$ — 2 pts

$\Rightarrow m^2 + 2m + 4 = 0$

$\Rightarrow m = \dfrac{-2 \pm \sqrt{-12}}{2} = -1 \pm \sqrt{3}\, i$ — 1 pt

$\Rightarrow$ The general solution is

$y = x^{-1}[c_1 \cos(\sqrt{3} \ln x) + c_2 \sin(\sqrt{3} \ln x)]$ — 2 pts

$y(1) = 4 \Rightarrow 4 = c_1$ — 1 pt

$y(e^{\frac{\pi}{2\sqrt{3}}}) = 0 \Rightarrow 0 = e^{-\frac{\pi}{2\sqrt{3}}}(0 + c_2)$

$\Rightarrow c_2 = 0$ — 1 pt

$\Rightarrow$ The solution is $y = 4x^{-1}\cos(\sqrt{3} \ln x)$ — 1 pt

9. (**17 points**) Solve $(D^2 + 5D + 4)y = 3 - 2x + 4e^x$

The auxiliary equation of the associated homogeneous equation is $m^2 + 5m + 4 = (m+1)(m+4) = 0$

so $y_c = c_1 e^{-x} + c_2 e^{-4x}$ (A) . (2 pts)

Now, since $D^2(3-2x) = 0$ and $(D-1)(4e^x) = 0$,

we apply the differential operator $D^2(D-1)$ to both sides of the DE. to obtain

$$D^2(D-1)(D^2+5D+4)y = 0$$

whose auxiliary eqn is $m^2(m-1)(m^2+5m+4) = 0$ — (6 pts)

Thus $y = c_1 e^{-x} + c_2 e^{-4x} + c_3 + c_4 x + c_5 e^x$ (B)

(A) and (B) gives a form of y_p: — (2 pts)

$$y_p = A + Bx + Ce^x$$

$$y_p' = B + Ce^x, \quad y_p'' = Ce^x$$

Substitute in the given DE, we get

$$Ce^x + 5(B + Ce^x) + 4(A + Bx + Ce^x) = 3 - 2x + 4e^x \text{ — (2 pts)}$$

$$\Rightarrow 10Ce^x + (5B + 4A) + 4Bx = 3 - 2x + 4e^x$$

$$\Rightarrow 10C = 4 \Rightarrow C = \frac{2}{5}, \text{ (1 pt)} \quad 5B + 4A = 3, \quad 4B = -2$$

$$\Rightarrow B = -\frac{1}{2} \text{ (1 pt) and } 4A = 3 + \frac{5}{2} = \frac{11}{2} \Rightarrow A = \frac{11}{8} \text{ — (1 pt)}$$

$$\Rightarrow y_p = \frac{11}{8} - \frac{1}{2}x + \frac{2}{5}e^x \text{ — (1 pt)}$$

$\Rightarrow$ The general solution is

$$y = y_c + y_p = c_1 e^{-x} + c_2 e^{-4x} + \frac{2}{5}e^x - \frac{1}{2}x + \frac{11}{8} \text{ — (1 pt)}$$