King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 202 - Exam I - Term 141

Duration: 120 minutes

Name: KEY ID Number:

Section Number: Serial Number:

Class Time: Instructor's Name:

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write neatly and eligibly. You may lose points for messy work.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 8 pages of problems (Total of 8 Problems)

Page Number	Points	Maximum Points
1		10
2		14
3		13
4		9
5		11
6		13
7		16
8		14
Total		100

1. a) (**4 points**) State the order of the given differential equation. Determine whether the equation is linear or nonlinear.

I. $x\dfrac{d^3y}{dx^3} - \left(\dfrac{dy}{dx}\right)^4 + y = 0$

order: 3 ; Nonlinear — 2pts

II. $(\sin\theta)\,y''' - (\cos\theta)\,y'' = 2$

order: 3 , Linear --- 2pts

b) (**6 points**) Find all values of m so that the function $y = x^m$ is a solution of the differential equation.

$$x^2y'' - 7xy' + 15y = 0$$

$y = x^m \Rightarrow y' = m x^{m-1} \Rightarrow y'' = m(m-1)\,x^{m-2}$ --- 2pts

That means, $x^2y'' - 7xy' + 15y = m(m-1)x^m - 7m x^m + 15 x^m = 0$ — 1pt

$\Rightarrow (m(m-1) - 7m + 15)\,x^m = 0$ -- 1pt

$\Rightarrow m^2 - 8m + 15 = 0 \Rightarrow (m-5)(m-3) = 0$ -- 1pt

$\Rightarrow m = 3$ & $m = 5$ — 1pt

2. Consider the differential equation $\frac{dy}{dx} - y^2 - y + 2 = 0$.

a) (**6 points**) Verify that $y = \frac{1+2ce^{3x}}{1-ce^{3x}}$ is a one-parameter family of solutions of the differential equation.

$$\frac{dy}{dx} = \frac{6Ce^{3x}(1-Ce^{3x}) + 3Ce^{3x}(1+2Ce^{3x})}{(1-Ce^{3x})^2}$$

$$= \frac{6Ce^{3x} - 6C^2e^{6x} + 3Ce^{3x} + 6C^2e^{6x}}{(1-Ce^{3x})^2} = \frac{9Ce^{3x}}{(1-Ce^{3x})^2} \quad \cdots \text{ 2pts}$$

Now, $LHS = \frac{dy}{dx} - y^2 - y + 2 = \frac{9Ce^{3x}}{(1-Ce^{3x})^2} - \frac{(1+2Ce^{3x})^2}{(1-Ce^{3x})^2} - \frac{1+2Ce^{3x}}{(1-Ce^{3x})} + 2$

$$= \frac{9Ce^{3x} - 1 - 4Ce^{3x} - 4C^2e^{6x} - (1+2Ce^{3x})(1-Ce^{3x}) + 2(1-Ce^{3x})^2}{(1-Ce^{3x})^2}$$

$$= \frac{5Ce^{3x} - 1 - 4C^2e^{6x} - 1 + Ce^{3x} - 2Ce^{3x} + 2C^2e^{6x} + 2 - 4Ce^{3x} + 2C^2e^{6x}}{(1-Ce^{3x})^2}$$

$= 0 = RHS$ --- 4pts

b) (**5 points**) Find two constant solutions of the differential equation.

Constant solutions means $y = C$ 1pt and so $\frac{dy}{dx} = 0$ -- 1pt

That means $\frac{dy}{dx} - y^2 - y + 2 = -C^2 - C + 2 = (2+C)(1-C) = 0$ 1pt

$\Rightarrow C = -2$ or $C = 1$ -- 2pts

Constant solutions are $y = -2$ or $y = 1$

c) (**3 points**) Find a singular solution of the differential equation.

$y = 1$ is part of the one-parameter family of soln. $(c = 0)$

$y = -2$ is not part =

So, $y = -2$ is a singular solution -- 3pts

3. a) (**7 points**) Does the initial-value problem $\frac{dy}{dx} = \sqrt{y^2-9}$, $y(2) = 5$ have a unique solution? **(Justify your answer)**

$f(x,y) = \sqrt{y^2-9}$ and so $\frac{\partial f}{\partial y} = \frac{y}{\sqrt{y^2-9}}$. --- 2 pts

f & $\frac{\partial f}{\partial y}$ are continuous if $y^2-9>0$, i.e., --- 2 pts

$y<-3$ or $y>3$. --- 1 pt

So, by the theorem, The IVP has a unique solution --- 2 pts

b) (**6 points**) Find a differentiable function f making the following differential equation exact.

$$3\left[f(x)+x^2\right]\sin 3y\,dx + \left(e^{3x} + \frac{x^3}{3} + x\right)\cos 3y\,dy = 0.$$

2 pts

DE is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, i.e.,

$9(f(x)+x^2)\cos 3y = (3e^{3x} + x^2 + 1)\cos 3y$ --- 2 pts

$9f(x)\cos 3y = 3e^{3x}\cos 3y - 8x^2\cos 3y + \cos 3y$

One possible solution is $f(x) = \frac{1}{3}e^{3x} - \frac{8}{9}x^2 + \frac{1}{9}$. --- 2 pts

4. (**9 points**) Solve the initial-value problem

$$xe^{2x+\cos y}\,dx + \sin y\,dy = 0$$

subject to $y(0) = \dfrac{\pi}{2}$.

$$x e^{2x} \cdot e^{\cos y}\, dx + \sin y\, dy = 0 \quad \text{--- 1pt} \quad * \, e^{-\cos y}$$

$$\Rightarrow x e^{2x}\, dx = -\sin y\, e^{-\cos y}\, dy \quad \text{--- 1pt}$$

separable

Integrate both sides

$$\int x e^{2x} dx = \frac{1}{2} x e^{2x} - \int \frac{1}{2} e^{2x} dx = \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + C \quad \text{--- 2pts}$$

$u = x \qquad dv = e^{2x} dx$

$du = dx \qquad v = \frac{1}{2} e^{2x}$

A one-parameter family of solutions is

$$\frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} = -e^{-\cos y} + C \quad \text{--- 2pts}$$

$$y(0) = \frac{\pi}{2} \Rightarrow -\frac{1}{4} = -1 + C \quad \text{--- 1pt} \quad \Rightarrow C = 3/4 \quad \text{--- 1pt}$$

Solution $\quad \dfrac{1}{2} x e^{2x} - \dfrac{1}{4} e^{2x} = -e^{-\cos y} + \dfrac{3}{4}$ --- 1pt

5. (**11 points**) Solve the linear differential equation

$$(y+1)\frac{dx}{dy} + (y+2)x = 2ye^{-y}$$

$$\Rightarrow \frac{dx}{dy} + \frac{y+2}{y+1}x = \frac{2y}{y+1}e^{-y} \quad \text{--- } 1\text{pt} \quad \text{linear in } x$$

$$p(y) = \frac{y+2}{y+1} \Rightarrow I.F. = e^{\int p(y)\,dy} = e^{\int \frac{y+2}{y+1}\,dy} \quad 1\text{pt} \quad 1\text{pt}$$

$$= e^{\int (1+\frac{1}{y+1})\,dy} = e^{y+\ln|y+1|} = (y+1)e^{y} \quad \text{--- } 2\text{pts}$$

Multiply by the I.F. to get

$$(y+1)e^{y}\frac{dx}{dy} + (y+2)e^{y}x = 2y \quad \text{--- } 2\text{pts}$$

$$\Rightarrow \frac{d}{dy}\left[(y+1)e^{y}x\right] = 2y \quad \text{--- } 2\text{pts}$$

Integrate

$$(y+1)e^{y}x = y^2 + c \quad \text{--- } 2\text{pts}$$

6. (**13 points**) Solve the differential equation

$$\frac{dy}{dx} = \frac{y^2 e^{xy^2} + 4x^3}{3y^2 - 2xy\, e^{xy^2}}$$

$$\Rightarrow \underbrace{(y^2 e^{xy^2} + 4x^3)}_{M} dx + (\underbrace{2xy e^{xy^2} - 3y^2}_{N}) dy = 0 \quad \text{--- 1 pt}$$

$$\Rightarrow \frac{\partial M}{\partial y} = 2y e^{xy^2} + 2xy^3 e^{xy^2} \text{ --- 1 pt} \equiv \frac{\partial N}{\partial x} = 2y e^{xy^2} + 2xy^3 e^{xy^2} \text{ --- 1 pt}$$

So, the DE is exact. --- 1 pt

$$\text{So, } \frac{\partial f}{\partial x} = y^2 e^{xy^2} + 4x^3 \text{ --- 1 pt} \Rightarrow f(x,y) = e^{xy^2} + x^4 + g(y) \text{ --- 2 pts}$$

$$\text{Now, } \frac{\partial f}{\partial y} = 2xy e^{xy^2} + g'(y) = 2xye^{xy^2} - 3y^2 = N \text{ --- 2 pts}$$

$$\Rightarrow g'(y) = -3y^2 \Rightarrow g(y) = -y^3 \text{ --- 2 pts}$$

$$\Rightarrow f(x,y) = e^{xy^2} + x^4 - y^3 \text{ --- 1 pt}$$

<u>Soln</u> $e^{xy^2} + x^4 - y^3 = C$ ---- 1 pt

7. Use an appropriate substitution to transform the differential equation

a) (**7 points**) $\frac{dy}{dx} = \frac{x^4}{y^2(x^2+y^2)}$ to a separable equation **(Do not solve)**

Homogeneous eqn. — 2 pts

Let $x = vy \Rightarrow dx = v\,dy + y\,dv$ — 2 pts

Substitute to get

$v^4y^4(v\,dy + y\,dv) - y^2(v^2y^2 + y^2)dy = 0$ — 1 pt

$\Rightarrow v^5y^4dy + v^4y^5dv - y^4v^2dy - y^4dy = 0$

$\Rightarrow v^4y^5dv = y^4(v^2+1-v^5)dy$ — 1 pt

$\Rightarrow \frac{v^4}{v^2-v^5+1}dv = \frac{dy}{y}$ — 1 pt separable

b) (**9 points**) $y^{-3/2}\frac{dy}{dx} + x^2y^{-1/2} = \frac{1}{x^2+1}$ to a linear equation **(Do not solve)**

This is a Bernoulli equation with $n = 3/2$ — 2 pts and it is clear after multiplying both sides by $y^{3/2}$ to get

$\frac{dy}{dx} + x^2y = \frac{1}{x^2+1}y^{3/2}$ — 1 pt

Substitute $u = y^{1-3/2} = y^{-1/2}$ — 1 pt or $u^{-2} = y$ — 1 pt to get

$\frac{dy}{dx} = -2u^{-3}\frac{du}{dx}$ — 1 pt and so the DE becomes

$-2u^{-3}\frac{du}{dx} + x^2u^{-2} = \frac{1}{x^2+1}u^{-3}$ — 1 pt

Now, multiply by $-\frac{1}{2}u^3$ to get

$\frac{du}{dx} - \frac{1}{2}x^2u = \frac{-1}{2(x^2+1)}$ — linear 2 pts

8. (**14 points**) The population of birds on a certain island grows at a rate proportional to the population present at time t. If the initial population of 1000 is increasing at a constant rate of 10% every year, then how long does it take for the population to tripple?

The DE is $\frac{dP}{dt} = kP$ where P is the population. 1pt

Also, $P(0) = 1000 = P_0$ & $P(1) = 1100$ --- 1pt

We need to find the time if $P = 3P_0$.

$$\frac{dP}{dt} = kP \Rightarrow \frac{dP}{P} = k\,dt \Rightarrow \ln|P| = kt + C$$

$$\Rightarrow P(t) = Ce^{kt} \text{ --- 2 pts}$$

1pt

Now, $P(0) = Ce^0 = 1000 \Rightarrow C = 1000 \Rightarrow P(t) = 1000e^{kt}$ 1pt

1pt

Also, $P(1) = 1100 \Rightarrow 1100 = 1000e^{k} \Rightarrow 1.1 = e^{k}$

$\Rightarrow k = \ln(1.1) \Rightarrow P(t) = 1000e^{t\ln(1.1)}$ --- 1pt

2 pts

The population tripples means $P(t) = 3P_0 = 3000$ and so --- 1pt

$$3000 = 1000e^{t\ln(1.1)}$$

$$\Rightarrow 3 = e^{t\ln(1.1)} \Rightarrow t\ln(1.1) = \ln 3 \text{ --- 2 pts}$$

$$\Rightarrow t = \frac{\ln 3}{\ln 1.1} \text{ --- 1 pt}$$