King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 201 - Exam I - Term 141

Duration: 120 minutes

Name: Key ID Number:

Section Number: Serial Number:

Class Time: Instructor's Name:

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write neatly and eligibly. You may lose points for messy work.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 8 pages of problems (Total of 8 Problems)

Page Number	Points	Maximum Points
1		10
2		10
3		10
4		15
5		15
6		10
7		10
8		20
Total		100

1. Let C be the parametric curve given by $x = 2 - \sec^2 t$, $y = \tan t$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

a) **(5 points)** Find a Cartesian equation of C.

Sol: $x = 2 - \sec^2 t = 1 - (\sec^2 t - 1)$ ③

$= 1 - \tan^2 t$ ①

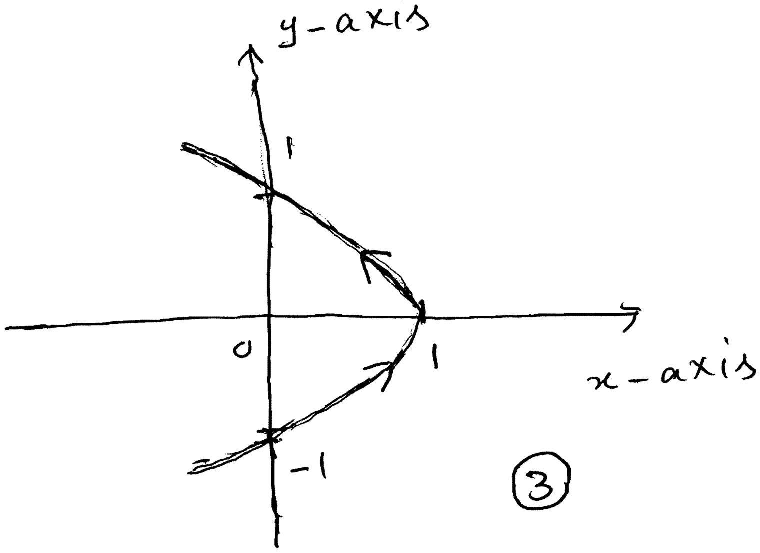
$= 1 - y^2$ ①

That is; $x = 1 - y^2$

b) **(5 points)** Sketch the graph of C and indicate with an arrow the direction in which the curve is traced as t increases.

t	x	y
$-\frac{\pi}{4}$	0	-1
0	1	0
$\frac{\pi}{4}$	0	1

②



③

2. **(10 points)** Find the length of the curve with parametric equations
$x = \dfrac{t^2}{2}$, $y = \dfrac{(2t+1)^{3/2}}{3}$, $0 \le t \le 4$.

Sol: Length $= \displaystyle\int_0^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ 2

$\dfrac{dx}{dt} = t$, $\dfrac{dy}{dt} = \dfrac{3}{2} \cdot \dfrac{(2t+1)^{\frac{3}{2}-1} \cdot 2}{3} = (2t+1)^{\frac{1}{2}}$ 2

$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{t^2 + (2t+1)} = \sqrt{(t+1)^2}$

$= |t+1|$ 2

Therefore

Length $= \displaystyle\int_0^4 |t+1|\, dt = \int_0^4 (t+1)\, dt$ 2

$= \left[\dfrac{t^2}{2} + t\right]_0^4 = 8 + 4 = 12.$ 2

3. **(10 points)** Find area S of the surface generated by revolving the curve about the y-axis.

$$x = 3t^2,\ y = 2t^3,\ 0 \le t \le 1$$

Sol: $S = 2\pi \int_0^1 x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ (2)

$\frac{dx}{dt} = 6t$, $\frac{dy}{dt} = 6t^2$, $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 36t^2 + 36t^4$

(3)

Therefore

$$S = 2\pi \int_0^1 3t^2 \sqrt{36t^2 + 36t^4}\, dt$$ (1)

$$= 2\pi \int_0^1 9t^2 \sqrt{1+t^2}\,(2t)dt$$

$$= 18\pi \int_0^1 t^2 \sqrt{1+t^2}\,(2t)\, dt$$ (1)

put $u = 1+t^2$. Then $du = 2tdt$,

$t^2 = u-1$, $t=0 \Rightarrow u=1$, $t=1 \Rightarrow u=2$.

(1)

Thus

$$S = 18\pi \int_1^2 (u-1)\, u^{\frac{1}{2}}\, du$$

$$= 18\pi \int_1^2 \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right) du = 18\pi \left[\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]_1^2$$

$$= 18\pi \left[\left(\frac{2}{5}\cdot 2^{\frac{5}{2}} - \frac{2}{3}\cdot 2^{\frac{3}{2}}\right) - \left(\frac{2}{5} - \frac{2}{3}\right)\right].$$ (2)

4. a) **(5 points)** Replace the following polar equation with the equivalent Cartesian

$$r = \cot 2\theta \ \csc\theta$$

Sol: $r = \cot 2\theta \csc\theta$

$$\Rightarrow r^2 = \cot^2 2\theta \csc^2\theta = \frac{\cos^2 2\theta}{\sin^2 2\theta}\cdot\frac{1}{\sin^2\theta} \quad (2)$$

$$\Rightarrow r^2\sin^2\theta = \frac{1-\sin^2 2\theta}{\sin^2 2\theta} = \frac{1}{\sin^2 2\theta} - 1 \quad (1)$$

$$\Rightarrow (r\sin\theta)^2 + 1 = \frac{1}{4\sin^2\theta\cos^2\theta} = \frac{r^4}{4(r\sin\theta)^2(r\cos\theta)^2} \quad (1)$$

$$\Rightarrow y^2+1 = \frac{(x^2+y^2)^2}{4x^2y^2} \Rightarrow 4x^2y^2(y^2+1) = (x^2+y^2)^2 \quad (1)$$

b) **(10 points)** Graph the polar curve: $r^2 = -\cos 2\theta$.

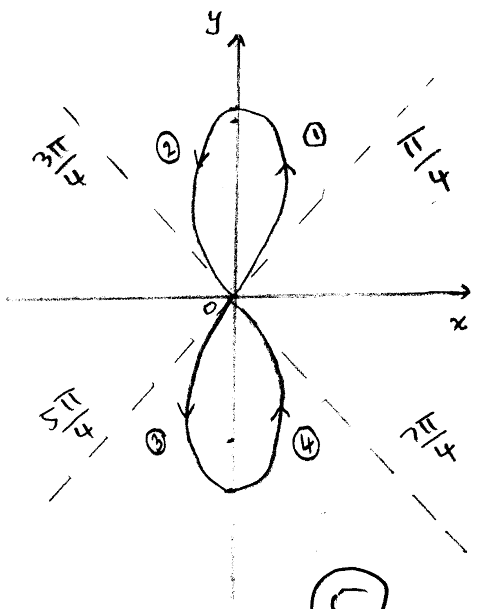
Sol: Table of values

θ	$r^2 = -\cos 2\theta$	r	θ	r^2	r
0	-1		$\frac{5\pi}{4}$	0	0
$\frac{\pi}{6}$	-0.5		$\frac{4\pi}{3}$	0.5	± 0.707
$\frac{\pi}{4}$	0	0	$\frac{3\pi}{2}$	1	± 1
$\frac{\pi}{3}$	0.5	± 0.707	$\frac{5\pi}{3}$	0.5	± 0.707
$\frac{\pi}{2}$	1	± 1	$\frac{7\pi}{4}$	0	0
$\frac{2\pi}{3}$	0.5	± 0.707	$\frac{11\pi}{6}$	-0.5	
$\frac{3\pi}{4}$	0	0	2π	-1	
$\frac{5\pi}{6}$	-0.5				
π	-1				
$\frac{7\pi}{6}$	-0.5				

(1) (2) (3) (4)

(5)

Graph



(5)

5. **(15 points)** Find the area of the region inside the circle $r = 4\sin\theta$ and below the line $r = 3\csc\theta$.

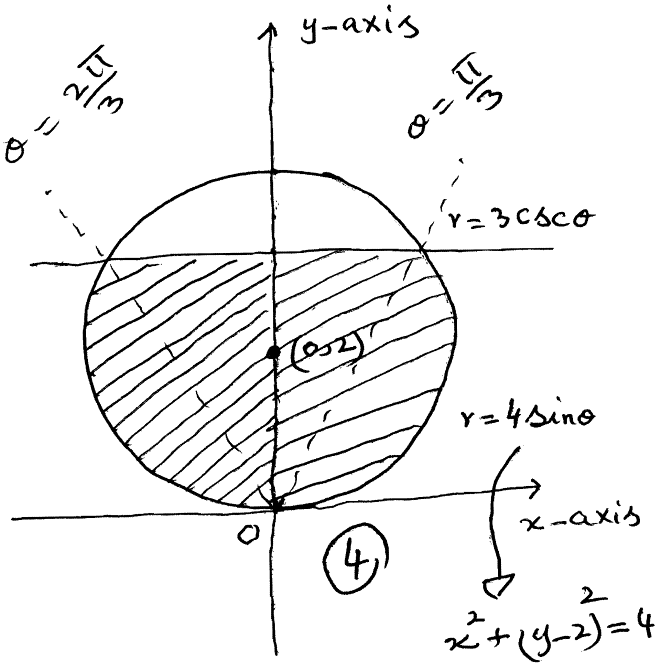
Sol: $3\csc\theta = r = 4\sin\theta$

$\Rightarrow \dfrac{3}{\sin\theta} = 4\sin\theta$

$\Rightarrow \sin^2\theta = \dfrac{3}{4}$

$\Rightarrow \sin\theta = \dfrac{\sqrt{3}}{2}$ (3)

$\Rightarrow \theta = \dfrac{\pi}{3}, \dfrac{2\pi}{3}$



Therefore $A = 4\pi - 2\displaystyle\int_{\pi/3}^{\pi/2} \frac{1}{2}\left(16\sin^2\theta - 9\csc^2\theta\right) d\theta$ (3)

$$= 4\pi - \int_{\pi/3}^{\pi/2} \left[16\frac{(1-\cos 2\theta)}{2} - 9\csc^2\theta\right] d\theta \quad (2)$$

$$= 4\pi - \int_{\pi/3}^{\pi/2} \left[8 - 8\cos 2\theta - 9\csc^2\theta\right] d\theta$$

$$= 4\pi - \Big[8\theta - 4\sin 2\theta + 9\cot\theta\Big]_{\pi/3}^{\pi/2} \quad (2)$$

$$= 4\pi - \left[(4\pi - 0 + 0) - \left(\frac{8\pi}{3} - 2\sqrt{3} + 3\sqrt{3}\right)\right]$$

$$= \frac{8\pi}{3} + \sqrt{3} \quad (1)$$

6. **(10 points)** Find the center and radius of the sphere

$$2x^2 + 2y^2 + 2z^2 + x + y + z = 9.$$

Sol: $2x^2 + 2y^2 + 2z^2 + x + y + z = 9$

$\Rightarrow x^2 + \frac{1}{2}x + y^2 + \frac{1}{2}y + z^2 + \frac{1}{2}z = \frac{9}{2}$ (2)

$\Rightarrow \left(x^2 + \frac{1}{2}x + \frac{1}{16}\right) + \left(y^2 + \frac{1}{2}y + \frac{1}{16}\right) + \left(z^2 + \frac{1}{2}z + \frac{1}{16}\right)$

$= \frac{9}{2} + \frac{3}{16}$ (3)

$\Rightarrow \left(x + \frac{1}{4}\right)^2 + \left(y + \frac{1}{4}\right)^2 + \left(z + \frac{1}{4}\right)^2 = \frac{75}{16} = \left(\frac{5\sqrt{3}}{4}\right)^2$ (3)

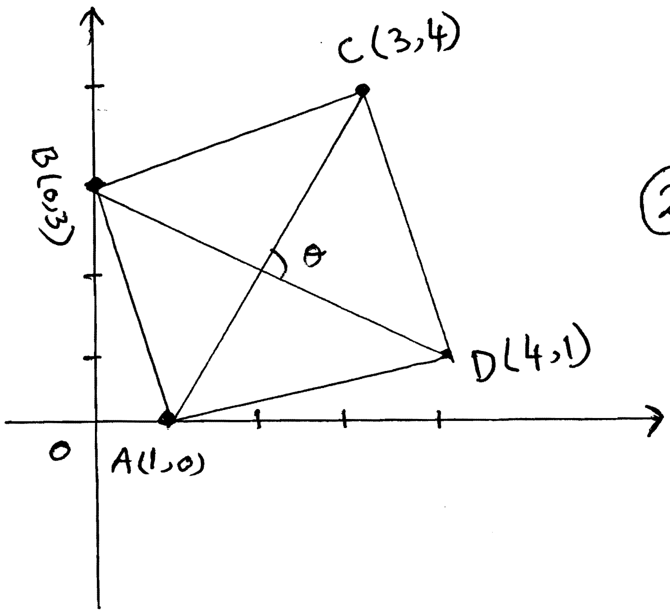
Centre $= \left(-\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right)$ (1)

radius $= \frac{5\sqrt{3}}{4}$ (1)

7. **(10 points)** Find the measures of the angles between the diagonals of the rectangle whose vertices are

$$A = (1, 0), B = (0, 3), C = (3, 4) \text{ and } D = (4, 1).$$

Sol:



2

$\vec{AC} = \langle 2, 4 \rangle$ 2, $\vec{BD} = \langle 4, -2 \rangle$ 2

$\vec{AC} \cdot \vec{BD} = 8 - 8 = 0$ 2

$$\cos\theta = \frac{\vec{AC} \cdot \vec{BD}}{|\vec{AC}||\vec{BD}|} = 0$$

$\Rightarrow \theta = \cos^{-1}(0) = \frac{\pi}{2}$. 2

The angle measures are all $\frac{\pi}{2}$.

8. a) **(10 points)** Find the volume of the parallelepiped determined by the vectors $\overrightarrow{AB}, \overrightarrow{AC}$, and $\overrightarrow{AD}$ where

$$A(2,0,-1), B(2,-1,1), C(1,3,2) \text{ and } D(3,-2,-1).$$

Sol: $\overrightarrow{AB} = \langle 2-2, -1-0, 1+1 \rangle = \langle 0,-1,2 \rangle$ (2)

$\overrightarrow{AC} = \langle 1-2, 3-0, 2+1 \rangle = \langle -1,3,3 \rangle$ (2)

$\overrightarrow{AD} = \langle 3-2, -2-0, -1+1 \rangle = \langle 1,-2,0 \rangle$ (2)

$$\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD}) = \begin{vmatrix} 0 & -1 & 2 \\ -1 & 3 & 3 \\ 1 & -2 & 0 \end{vmatrix} = -3 + 2(2-3) = -5$$ (2)

$$\text{Volume} = |\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD})| = |-5| = 5$$ (2)

b) **(10 points)** Find two unit vectors orthogonal to both the vectors $\vec{a} = \langle 1,1,-1 \rangle$ and $\vec{b} = \langle -1,2,2 \rangle$.

Sol: $$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ -1 & 2 & 2 \end{vmatrix} = (2+2)\vec{i} - (2-1)\vec{j} + (2+1)\vec{k} = 4\vec{i} - \vec{j} + 3\vec{k}$$ (4)

$$\Rightarrow |\vec{a} \times \vec{b}| = \sqrt{(4)^2 + (-1)^2 + (3)^2} = \sqrt{16+1+9} = \sqrt{26}$$ (2)

The required unit vectors are

$$\vec{u}_1 = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \frac{1}{\sqrt{26}}(4\vec{i} - \vec{j} + 3\vec{k})$$ (2)

and $$\vec{u}_2 = \frac{-(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|} = \frac{1}{\sqrt{26}}(-4\vec{i} + \vec{j} - 3\vec{k}).$$ (2)