King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 201 - Final Exam - Term 141

Duration: 180 minutes

Name: Key　　　　ID Number: ______

Section Number: ______　　　　Serial Number: ______

Class Time: ______　　　　Instructor's Name: ______

Instructions:

1. Calculators and Mobiles are not allowed.
2. This exam consists of two parts: Written and Multiple Choice.
3. Show all your work. No points for answers without justification.

Parts	Points	Maximum Points
MCQ		70
Written		70
Total		140

1. The area of the surface generated by revolving the curve $x = \cos^2 t, y = \sin^2 t \left(0 \leq t \leq \frac{\pi}{2}\right)$ about the x-axis is

(a) $\sqrt{2}\,\pi$

(b) $\frac{\pi}{\sqrt{2}}$

(c) $2\,\pi$

(d) $2\sqrt{2}\,\pi$

(e) 4π

2. Slope of the tangent line to the curve $r = 2 + 2\cos\theta$ at $\theta = \frac{\pi}{6}$ is

(a) -1

(b) 1

(c) $\frac{1}{2}$

(d) $-\frac{1}{2}$

(e) $\frac{2}{3}$

3. The polar coordinates $(r, \theta), 0 < \theta < 4\pi$ and $r < 0$, of the point $(1, -1)$ in Cartesian coordinates is

(a) $\left(-2, \frac{11\pi}{4}\right)$

(b) $\left(-2, \frac{7\pi}{4}\right)$

(c) $\left(-2, \frac{9\pi}{4}\right)$

(d) $(-2, \pi)$

(e) $\left(-2, \frac{\pi}{4}\right)$

4. The length of the polar curve $r = 3 + 3\sin\theta$ is

(a) 24

(b) 22

(c) 26

(d) 13

(e) 11

5. The vector from the point $(1, 2, -3)$ to the centre of the sphere $x^2+y^2+z^2-kx+3y-lz = 1$ is given by $< 3, h, -2 >$. Then the value of hkl is

(a) 280

(b) 290

(c) 208

(d) -200

(e) 360

6. The area of the triangle with vertices $P(1, 2, 3)$, $Q(-1, 0, 4)$, $R(1, -2, -3)$ is equal to

(a) $2\sqrt{29}$

(b) $3\sqrt{29}$

(c) $\sqrt{29}$

(d) $4\sqrt{29}$

(e) $\frac{3}{2}\sqrt{29}$

7. $\lim_{(x,y)\to(0,0)} \dfrac{x^2+y}{\sqrt{4+2x^2+2y}-2}$ is equal to

(a) 2

(b) 1

(c) 0

(d) $\dfrac{3}{2}$

(e) $-\dfrac{1}{2}$

8. If $R = \{(x,y) : \ 1 \le x \le 2, \ 0 \le y \le 4\}$, then $\displaystyle\int\int_R \frac{\sqrt{y}}{x^2}\, dA$ is equal to

(a) $\dfrac{8}{3}$

(b) $\dfrac{4}{3}$

(c) $\dfrac{16}{3}$

(d) 0

(e) $-\dfrac{4}{3}$

9. The distance from the point $(1,1,5)$ to the line

$$L: x = 1+t, y = 3-t, z = 2t$$

is

(a) $\sqrt{5}$

(b) 5

(c) $\sqrt{7}$

(d) $\sqrt{2}$

(e) 3

10. $\int_0^1 \int_1^{\sqrt{e}} \int_1^e \frac{xe^x \ln y (\ln z)^2}{z} \, dz\, dy\, dx$ is equal to

(a) $\frac{2-\sqrt{e}}{6}$

(b) $\frac{2-\sqrt{e}}{3}$

(c) $\frac{2-\sqrt{e}}{4}$

(d) $\frac{\sqrt{e}-2}{3}$

(e) $\frac{2-e}{3}$

11. Let $z = \tan^{-1}\left(\frac{u}{v}\right)$ where $u(x,y) = 2x + y$ and $v(x,y) = 3x - y$. Then $\frac{\partial z}{\partial y}|_{(1,1)}$ is

(a) $\frac{5}{13}$

(b) 0

(c) -1

(d) $\frac{3}{13}$

(e) $\frac{5}{11}$

12. The linearization $L(x,y)$ of the function $f(x,y) = x^2y + \sqrt{x^2+y^2}$ at the point $(1,0)$ is

(a) $x + y$

(b) $x - y$

(c) $x + y + 1$

(d) $x + y + 2$

(e) $y - x$

13. The equation of the sphere with center at $(1,2,3)$ and tangent to the line $x = 2 + t, y = 3 - t, z = 4$ is

(a) $x^2 + y^2 + z^2 - 2x - 4y - 6z + 11 = 0$

(b) $x^2 + y^2 + z^2 - 4x - 2y - 6z + 11 = 0$

(c) $x^2 + y^2 + z^2 - 6x - 4y - 2z + 11 = 0$

(d) $x^2 + y^2 + z^2 - 2x - 6y - 4z + 11 = 0$

(e) $x^2 + y^2 + z^2 - 2x - 5y - 4z + 11 = 0$

14. For $Z = \ln(x^2 + y^2) + 2\tan^{-1}\left(\frac{y}{x}\right)$, $Z_{xx} + Z_{yy}$ is equal to

(a) 0

(b) -1

(c) $\frac{1}{2}$

(d) 1

(e) $-\frac{1}{2}$

Part II: Written Questions

Instructions for written Questions

1. This part has 6 questions
2. Points may be deducted for poor presentation
3. No credits will be given to wrong steps

Question Number	Obtained Points	Maximum Points
15		10
16		12
17		12
18		12
19		12
20		12
Total		70

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15. **(10 points)** Find the local extreme values (if any) of the function

$$f(x,y) = \ln(x+y) + x^2 - y.$$

Sol : $f_x(x,y) = \frac{1}{x+y} + 2x$, $f_y(x,y) = \frac{1}{x+y} - 1$

(2 pts)

Critical points

$f_x(x,y) = 0 = f_y(x,y) \Rightarrow \frac{1}{x+y} + 2x = 0$ and $\frac{1}{x+y} = 1.$

Therefore $1 + 2x = 0 \Rightarrow x = -\frac{1}{2}$

and $\frac{1}{x+y} = 1 \Rightarrow y = 1 - x = 1 + \frac{1}{2} = \frac{3}{2}.$

We have only one critical point $(-\frac{1}{2}, \frac{3}{2})$.

(3 pts)

Now $f_{xx}(x,y) = -\frac{1}{(x+y)^2} + 2$, $f_{yy}(x,y) = -\frac{1}{(x+y)^2}$,

$f_{xy}(x,y) = -\frac{1}{(x+y)^2}$.

$\Rightarrow f_{xx}(-\frac{1}{2}, \frac{3}{2}) = 1$, $f_{yy}(-\frac{1}{2}, \frac{3}{2}) = -1 = f_{xy}(-\frac{1}{2}, \frac{3}{2})$.

(3 pts)

$\Rightarrow f_{xx}(-\frac{1}{2}, \frac{3}{2}) f_{yy}(-\frac{1}{2}, \frac{3}{2}) - f_{xy}^2(-\frac{1}{2}, \frac{3}{2}) = -1 - 1 = -2 < 0$

Hence $(-\frac{1}{2}, \frac{3}{2})$ is a saddle point for f

and no local extreme values. (2 pts)

16. **(12 points)** Use the Lagrange Multiplier method to find extrema of $f(x, y, z) = z - x^2 - y^2$ subject to the constraints $x + y + z = 1$ and $x^2 + y^2 = 4$.

<u>Sol</u>: By Lagrange multiplier method

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z) \text{ where}$$

$$g(x,y,z) = x+y+z-1 = 0$$
$$h(x,y,z) = x^2+y^2-4 = 0$$

(2 pts)

$$\Rightarrow \langle -2x, -2y, 1\rangle = \lambda\langle 1,1,1\rangle + \mu\langle 2x, 2y, 0\rangle$$
$$= \langle \lambda + 2x\mu, \lambda + 2y\mu, \lambda\rangle$$

(2 pts)

$$\Rightarrow \left.\begin{array}{l} -2x = \lambda + 2x\mu \\ -2y = \lambda + 2\mu y \\ \lambda = 1 \end{array}\right] \Rightarrow \left.\begin{array}{l} 2x(1+\mu) = -\lambda = -1 \\ 2y(1+\mu) = -\lambda = -1 \end{array}\right] \Rightarrow x = y$$

(2 pts)

Now $x^2+y^2-4=0 \Rightarrow 2x^2 = 4 \Rightarrow x = \pm\sqrt{2} = y$.

(2 pts)

$x = y = \sqrt{2} \Rightarrow z = 1 - \sqrt{2} - \sqrt{2} = 1 - 2\sqrt{2}$

$x = y = -\sqrt{2} \Rightarrow z = 1 + \sqrt{2} + \sqrt{2} = 1 + 2\sqrt{2}$

(2 pts)

Now $f(\sqrt{2}, \sqrt{2}, 1-2\sqrt{2}) = -3 - 2\sqrt{2}$ (local minimum)

and $f(-\sqrt{2}, -\sqrt{2}, -3+2\sqrt{2}) = -3 + 2\sqrt{2}$ (local maximum)

(2 pts)

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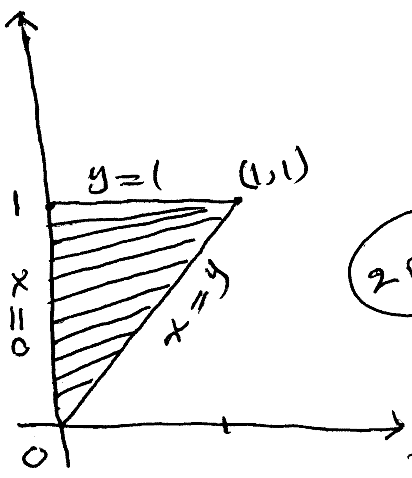
17. **(12 points)** Find the absolute maximum and minimum of the function $f(x, y) = x^2 + xy + y^2 - 6x$ defined on the closed triangle $x = 0, y = 1, y = x$.

Sol : Inside the domain



$f(x,y) = x^2 + xy + y^2 - 6x$

$\Rightarrow f_x(x,y) = 2x + y - 6 = 0 \longrightarrow$ ⓐ

and $f_y(x,y) = x + 2y = 0 \longrightarrow$ ⓑ (2pts)

Solving ⓐ and ⓑ, we get that $(x,y) = (4,-2)$ which is not inside the domain.

Hence $(4,-2)$ is not the critical point of f.

On the boundary $y = 1$: (1 pt)

$g(x) = f(x,1) = x^2 - 5x + 1$ on interval $[0,1]$.

$\Rightarrow g'(x) = 2x - 5 = 0 \Rightarrow x = \frac{5}{2}$ is not in the domain of $g \Rightarrow$ Not a critical point.

Now $g(0) = 1$, $g(1) = -3$ (2pts)

On the boundary $x = 0$:

$h(y) = f(0,x) = y^2$ on the interval $[0,1]$.

$h'(y) = 2y = 0$ not a critical point , $h(0) = 0, h(1) = 1$.

On the boundary $x = y$: $K(x) = f(x,x) = 3x^2 - 6x$ (2pts)

$\Rightarrow K'(x) = 6x - 6 = 0 \Rightarrow x = 1$, not a c.p. $K(0) = 0$, $K(1) = -3$. (2pts)

Absolute max. of $f = 1$ at $(0,0)$, Absolute max. of $f(x,y) = -3$ at $(1,1)$. (1 pt)

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18. **(12 points)** Sketch the region of integration for the integral $\int_0^3 \int_{\sqrt{\frac{x}{3}}}^1 \sin(\pi y^3)\, dy\, dx$ and evaluate it.

Sol:

$$\int_0^3 \int_{\sqrt{\frac{x}{3}}}^1 \sin(\pi y^3)\, dy\, dx$$

$$= \int_0^1 \int_0^{3y^2} \sin(\pi y^3)\, dx\, dy$$

(4pts)

$$= \int_0^1 \sin(\pi y^3) \Big[x\Big]_0^{3y^2} dy$$

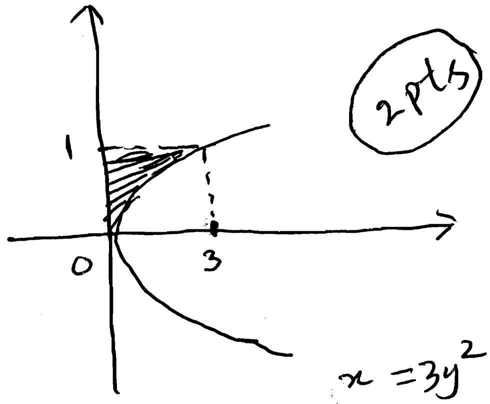
$$= \int_0^1 3y^2 \sin(\pi y^3)\, dy$$

$$= -\frac{1}{\pi}\Big[\cos(\pi y^3)\Big]_0^1$$

$$= -\frac{1}{\pi}\Big[\cos\pi - \cos 0\Big]$$

(4pts)

$$= \frac{2}{\pi}$$



(2pts)

$0 \leq x \leq 3$

$\sqrt{\frac{x}{3}} \leq y \leq 1$

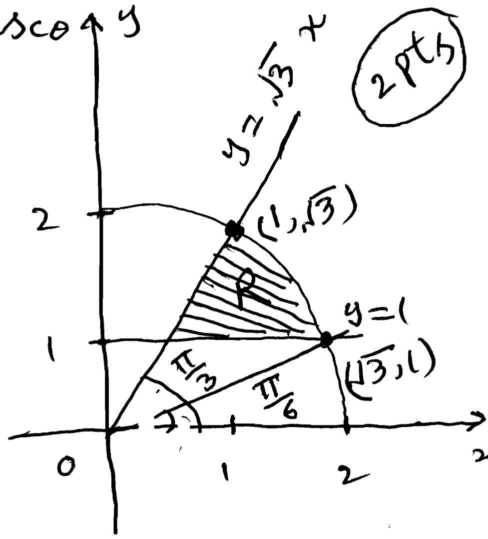
⇓ reversing the order

$0 \leq x\ 3y^2$

$0 \leq y \leq 1$

(2pts)

19. **(12 points)** Using polar integration, find the area of the region R in the xy-plane enclosed by the circle $x^2 + y^2 = 4$, above the line $y = 1$ and below the line $y = \sqrt{3}\,x$.



Sol: $y = 1 \Rightarrow r\sin\theta = 1 \Rightarrow r = \csc\theta$

and $x^2+y^2 = 4 \Rightarrow r^2 = 4 \Rightarrow r = 2$

Therefore $\csc\theta \le r \le 2$. (2pts)

The line $y = \sqrt{3}\,x$ has slope $\tan\theta = \sqrt{3}$, therefore

$\theta = \frac{\pi}{3}$.

The radial line from the origin through the point $(\sqrt{3},1)$ has slope $\tan\theta = \frac{1}{\sqrt{3}}$

Therefore $\theta = \frac{\pi}{6}$. That is, $\frac{\pi}{6} \le \theta \le \frac{\pi}{3}$ (2pts)

$$\text{Area} = \int_{\pi/6}^{\pi/3}\int_{\csc\theta}^{2} r\,dr\,d\theta \quad \text{(2pts)}$$

$$= \int_{\pi/6}^{\pi/3} \left[\tfrac{1}{2}r^2\right]_{r=\csc\theta}^{2} d\theta = \int_{\pi/6}^{\pi/3} \tfrac{1}{2}(4-\csc^2\theta)\,d\theta \quad \text{(2pts)}$$

$$= \tfrac{1}{2}\left[4\theta + \cot\theta\right]_{\pi/6}^{\pi/3} = \tfrac{1}{2}\left[\left(\tfrac{4\pi}{3}+\tfrac{1}{\sqrt{3}}\right) - \left(\tfrac{4\pi}{6}+\sqrt{3}\right)\right]$$

$$= \frac{\pi - \sqrt{3}}{3} \quad \text{(2pts)}$$

20. **(12 points)** If $B = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$ is the solid unit ball, then evaluate $\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, dv$

Sol: Since the boundary of B is a sphere, we use spherical coordinates:

$$B = \{(\rho, \theta, \phi) : 0 \leq \rho \leq 1,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \pi\}$$ (3 pts)

Also $x^2 + y^2 + z^2 = \rho^2$. (1 pt)

Therefore

$$\iiint_B e^{(x^2+y^2+z^2)^{\frac{3}{2}}}\, dv = \int_0^{\pi}\int_0^{2\pi}\int_0^{1} e^{(\rho^2)^{\frac{3}{2}}} \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$ (4 pts)

$$= \int_0^{\pi} \sin\phi\, d\phi \int_0^{2\pi} d\theta \int_0^{1} \rho^2 e^{\rho^3}\, d\rho$$

$$= \Big[-\cos\phi\Big]_0^{\pi} \Big[\theta\Big]_0^{2\pi} \Big[\tfrac{1}{3} e^{\rho^3}\Big]_0^{1}$$

$$= \frac{2\pi}{3}(-\cos\pi + \cos 0)(e-1)$$ (3 pts)

$$= \frac{4\pi}{3}(e-1)$$ (1 pt)