King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
STAT-361 Operations Research I [1]
Midterm Exam
Three Problems, April 3^{rd}, 2014 [2]

Problem 1 (25 pts)

Given the following pair of linear programs:

$$\begin{array}{lll} \max\limits_{x,y} & z = 2x + 3y \\ \text{s.t.} & x + 2y \le 5, \\ & 2x + y \le 4, \\ & x, y \ge 0 \end{array} \qquad \begin{array}{lll} \min\limits_{\alpha,\beta} & \gamma = 5\alpha + 4\beta \\ \text{s.t.} & -\alpha - 2\beta \le -2 \rightarrow \alpha + 2\beta \geq 2 \\ & 2\alpha + \beta \ge 3 \\ & \alpha, \beta \ge 0 \end{array}$$

(a) What is the relation between these two linear programs. (5pts)

(b) Write the linear complementary conditions corresponding to these two linear programs. (10pts)

(c) Solve both programs graphically and give their optimal solutions.(10pts)

5 (a) Primal & Dual.

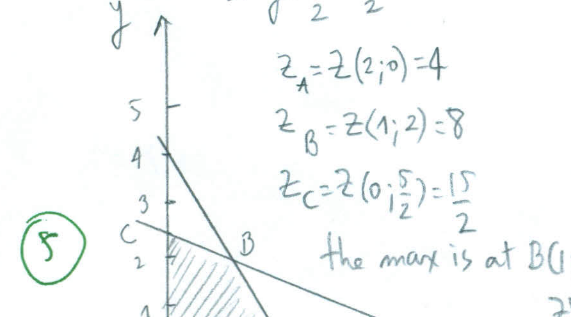
10 (b) $\alpha(5 - x - 2y) = 0$ & $\beta(4 - 2x - y) = 0$.

(c)

$f_1: y = 4 - 2x$
$f_2: y = \frac{5}{2} - \frac{1}{2}x$

$z_A = z(2;0) = 4$
$z_B = z(1;2) = 8$
$z_C = z(0;\frac{5}{2}) = \frac{15}{2}$

the max is at $B(1;2)$. $z^* = 8$.



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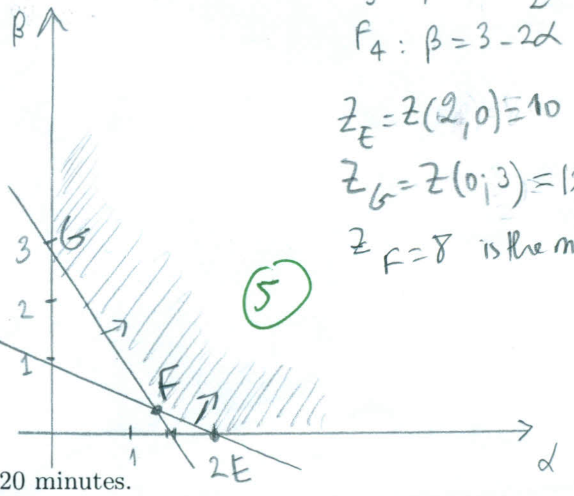
$F_3: \beta = 1 - \frac{\alpha}{2}$
$F_4: \beta = 3 - 2\alpha$

$z_E = z(2,0) = 10$
$z_G = z(0;3) = 12$
$z_F = 8$ is the min.



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[1] Dr. Slim Belhaiza (c)
[2] This is NOT an open book exam. The exam lasts 120 minutes.

$B: \begin{cases} 2x + y = 4 \\ x + 2y = 5 \end{cases} \rightarrow 3y = 6 \rightarrow y = 2$ & $x = 1$. $\Rightarrow B(1;2)$

$F: \begin{cases} \beta = 1 - \frac{\alpha}{2} = 3 - 2\alpha \end{cases}$
$\Rightarrow \frac{3}{2}\alpha = 2$
$\Rightarrow \alpha = \frac{4}{3};\ \beta = \frac{1}{3}$
$F(\frac{4}{3};\frac{1}{3})$.

Problem 2 **(40 pts)**

Consider the following linear program:

$$
\begin{aligned}
\max_{x_1,x_2,x_3} \quad & 3x_1 + 2x_2 + x_3 \\
\text{s.t.} \quad & x_1 + x_2 + 2x_3 \leq 3, \\
& x_1 - x_2 + x_3 \geq 2, \\
& 2x_1 + x_2 + x_3 \leq 4, \\
& x_1, x_2, x_3 \geq 0.
\end{aligned}
$$

(a) Solve the linear program using the Primal Simplex algorithm.(20pts)

(b) Solve the linear program using the Dual Simplex algorithm.(20pts)

(b) the standard form: $\max\ 3x_1 + 2x_2 + x_3$

$$\text{s.t.} \begin{cases} x_1 + x_2 + 2x_3 \leq 3 \\ x_1 - x_2 + x_3 \geq 2 \\ 2x_1 + x_2 + x_3 \leq 4 \\ (x_1; x_2; x_3) \geq 0 \end{cases} \Rightarrow \max\ 3x_1 + 2x_2 + x_3 \quad \text{s.t.} \begin{cases} x_1 + x_2 + 2x_3 + e_1 = 3 \\ -x_1 + x_2 - x_3 + e_2 = -2 \\ 2x_1 + x_2 + x_3 + e_3 = 4 \\ (x_1; x_2; x_3) \geq 0 \end{cases}$$

Dual Simplex

Tab1.

	3	2	1	0	0	0		
	x_1	x_2	x_3	e_1	e_2	e_3		
e_1	1	1	2	1	0	0	3	
e_2	-1	1	-1	0	1	0	-2	$< 0 \rightarrow$ Leaves
e_3	2	1	1	0	0	1	4	
	3	2	1	0	0	0		

x_3 Enters.

Tab2.

		3	2	1					
		x_1	x_2	x_3	e_1	e_2	e_3		
	e_1	-1	3	0	1	2	0	-1	$< 0 \rightarrow$ leaves
1	x_3	1	-1	1	0	-1	0	2	
	e_3	1	2	0	0	0	1	2	
		2	3	0	0	1	0		

x_1 Enters

Tab3.

		3	2	1	0	0	0	
		x_1	x_2	x_3	e_1	e_2	e_3	
3	x_1	1	-3	0	-1	-2	0	1
1	x_3	0	2	1	1	1	0	7
0	e_3	0	5	0	1	2	1	7 $\rightarrow$ Leaves
		0	9	0	2	5	0	

We now switch to the primal-simplex

x_2 Enters, e_3 leaves.

Tab4.

		3	2	1	0	0	0	
		x_1	x_2	x_3	e_1	e_2	e_3	
3	x_1	1	0	0	$-2/5$	$-\frac{4}{5}$	$3/5$	$8/5$
1	x_3	0	0	1	$3/5$	$1/5$	$-2/5$	$3/5$
2	x_2	0	1	0	$1/5$	$2/5$	$1/5$	$1/5$
		0	0	0	$2/5$	$-7/5$	$-9/5$	$z = \frac{29}{5}$

e_1 enters, x_3 leaves.

		3	2	1	0	0	0	
		x_1	x_2	x_3	e_1	e_2	e_3	
3	x_1	1	0	$2/3$	0	$-1/3$	$1/3$	2
0	e_1	0	0	$5/3$	1	$1/3$	$-2/3$	1
2	x_2	0	1	$-1/3$	0	$2/3$	$1/3$	0
		0	0	$-1/3$	0	$-1/3$	$-5/3$	$z^* = 6$

the optimal solution is

$x_1^* = 2;\ x_2^* = 0;\ e_1^* = 1$

$z^* = 6.$

(a) Primal Simplex:

Standard Form:

$$\max\ 3x_1 + 2x_2 + x_3 - Ma$$

s.t.

$$\begin{cases} x_1 + x_2 + 2x_3 + e_1 = 3 \\ x_1 - x_2 + x_3 - e_2 + a = 2 \\ 2x_1 + x_2 + x_3 + e_3 = 4 \end{cases}$$

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$x_1 ; x_2 ; x_3 \geq 0,$

$a \geq 0,\ e_1, e_2, e_3 \geq 0.$

Tab1.

		3 x_1	2 x_2	1 x_3	0 e_1	0 e_2	0 e_3	$-M$ a	
0	e_1	1	1	2	1	0	0	0	3
$-M$	a	1	-1	1	0	-1	0	1	2
0	e_3	2	1	1	0	0	1	0	4
		$3+M$	$2-M$	$1+M$	0	$-M$	0	0	

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x_1 enters, a leaves.

Tab2.

		3 x_1	2 x_2	1 x_3	0 e_1	0 e_2	0 e_3	$-M$ a	
0	e_1	0	2	1	1	1	0	-1	1
3	x_1	1	-1	1	0	-1	0	1	2
0	e_3	0	3	-1	0	2	1	-2	0 → leaves
		0	$\boxed{5}$	-2	0	3	0	$3-M$	

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Tab3. x_2 enters, e_3 leaves.

		3 x_1	2 x_2	1 x_3	0 e_1	0 e_2	0 e_3	$-M$ a	
0	e_1	0	0	$5/3$	1	$-1/3$	$-2/3$	$1/3$	1
3	x_1	1	0	$2/3$	0	$-1/3$	$1/3$	$1/3$	2
2	x_2	0	1	$-1/3$	0	$2/3$	$1/3$	$-2/3$	0
		0	0	$-1/3$	0	$-1/3$	$-1/3$	$-M$	$Z^* = 6$

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the optimal solution:

$$x_1^* = 2 ;\ x_2^* = 0 ;\ e_1^* = 1$$

$$Z^* = 6.$$

Problem 3 (35 Points)

Given the following linear program:

$$\begin{aligned} \max_{x_1,x_2,x_3} \quad & 3x_1 + 5x_2 + 2x_3 \\ \text{s.t.} \quad & 2x_1 + 4x_2 + x_3 \leq 7, \\ & 3x_1 + 2x_2 + x_3 \leq 4, \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

(a) Write the standard form corresponding to the linear program.(5pts)

(b) Solve the linear program using the Revised Simplex method.(30pts)

(a) $\max_{x_1;x_2;x_3} \; 3x_1+5x_2+2x_3$

s.t. $\begin{cases} 2x_1+4x_2+x_3+e_1=7 \\ 3x_1+2x_2+x_3+e_2=4 \\ x_1;x_2;x_3\geq 0. \end{cases}$

5

(b) • Base: $(e_1;e_2) \to B^{-1}=I$; $N=\begin{pmatrix}2&4&1\\3&2&1\end{pmatrix}$; $c_B^t=(0,0)$; $c_N^t=(3\ 5\ 2)$

$Rc_N^t = c_N^t - c_B^t B^{-1} N = (3\ 5\ 2) - (0\ 0)\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}2&4&1\\3&2&1\end{pmatrix}$

$= (3\ 5\ 2) > 0$

4

$\to x_1$ enters, e_2 leaves.

• Base: $(e_1, x_1) \to B=\begin{pmatrix}1&2\\0&3\end{pmatrix} \to |B|=3 \to B^{-1}=\frac{1}{3}\begin{pmatrix}3&-2\\0&1\end{pmatrix}$ $\begin{pmatrix}1&2\\0&3\end{pmatrix} = \frac{1}{3}\begin{pmatrix}3&0\\0&3\end{pmatrix}$

$Rc_N^t = c_N^t - c_B^t B^{-1} N = (5\ 2\ 0) - (0\ 3)\begin{pmatrix}1&-2/3\\0&1/3\end{pmatrix}\begin{pmatrix}4&1&0\\2&1&1\end{pmatrix}$

$= (5\ 2\ 0) - (0\ 1)\begin{pmatrix}4&1&0\\2&1&1\end{pmatrix}$

$= (5\ 2\ 0) - (2\ 1\ 1)$

$= (3\ 1\ -1)$

7

$X_B = B^{-1} b = \frac{1}{3}\begin{pmatrix}3&-2\\0&1\end{pmatrix}\begin{pmatrix}7\\4\end{pmatrix} = \frac{1}{3}\begin{pmatrix}13\\4\end{pmatrix} = \begin{pmatrix}13/3\\4/3\end{pmatrix}$

$B^{-1}.N = \frac{1}{3}\begin{pmatrix}3&-2\\0&1\end{pmatrix}\begin{pmatrix}4&1&0\\2&1&1\end{pmatrix} = \begin{pmatrix}1&-2/3\\0&1/3\end{pmatrix}\begin{pmatrix}4&1&0\\2&1&1\end{pmatrix} = \begin{pmatrix}\boxed{\begin{matrix}8/3\\2/3\end{matrix}} & \begin{matrix}1/3\\1/3\end{matrix} & \begin{matrix}-2/3\\1/3\end{matrix}\end{pmatrix}$

3

x_2 enters, $\min\left[\frac{13}{8} ; \frac{4}{2}\right] = \frac{13}{8} \Rightarrow e_1$ leaves.

• Base: $(x_2; x_1) \to B=\begin{pmatrix}4&2\\2&3\end{pmatrix} \to |B|=8 \to B^{-1}=\frac{1}{8}\begin{pmatrix}3&-2\\-2&4\end{pmatrix}$ $\begin{pmatrix}4&2\\2&3\end{pmatrix} = \begin{pmatrix}8/8&0\\0&8/8\end{pmatrix} = I$

$$RC_N^t = C_N^t - C_B^t B^{-1} N = (2\ \ 0\ \ 0) - (5\ \ 3)\begin{pmatrix} 3/8 & -1/4 \\ -1/4 & 1/2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= (2\ 0\ 0) - (9/8\ \ 1/4)\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = (2\ 0\ 0) - (7/8\ \ 9/8\ \ 1/4)$$

$$= \left(\frac{9}{8}\ \ -\frac{9}{8}\ \ -\frac{1}{4} \right)$$

7

$$x_B = B^{-1}\cdot b = \begin{pmatrix} 3/8 & -1/4 \\ -1/4 & 1/2 \end{pmatrix}\begin{pmatrix} 7 \\ 4 \end{pmatrix} = \begin{pmatrix} 13/8 \\ 1/4 \end{pmatrix}$$

$$B^{-1} N = \begin{pmatrix} 3/8 & -1/4 \\ -1/4 & 1/2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \boxed{\begin{matrix} 1/8 \\ 1/4 \end{matrix}} & \begin{matrix} 3/8 \\ -1/4 \end{matrix} & \begin{matrix} -1/4 \\ 1/2 \end{matrix} \end{pmatrix}$$

x_3 enters, $\min\{13, 1\} = 1 \Rightarrow x_1$ leaves.

- Base: $(x_2 ; x_3) \rightarrow B = \begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix} \rightarrow |B| = 2 \rightarrow B^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = I$

$$RC_N^t = C_N^t - C_B^t B^{-1}\cdot N = (3\ \ 0\ \ 0) - (5\ \ 2)\begin{pmatrix} 1/2 & -1/2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

7

$$= (3\,0\,0) - (1/2\ \ 3/2)\begin{pmatrix} 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} = (3\ 0\ 0) - (11/2\ \ 1/2\ \ 3/2)$$

$$= \left(-\frac{5}{2}\ \ -1/2\ \ -3/2 \right) < 0.$$

$$x_B = B^{-1} b = \begin{pmatrix} 1/2 & -1/2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 7 \\ 4 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 1 \end{pmatrix}.$$

5

$$z^* = \frac{15}{2} + 2 = \frac{19}{2}.$$