"Make-up" First Major Exam

Monday April 14, 2014

Name: _Khalid_____   ID #: _____

1) A student has to answer 7 out of 10 questions in an examination.   (1 pt.)
   a) How many choices does he have?

$$\binom{10}{7} = 120$$

   b) How many choices, if he must answer at least 3 of the first 5 questions?   (3 pts.)

$$\binom{5}{3}\binom{5}{4} + \binom{5}{4}\binom{5}{3} + \binom{5}{5}\binom{5}{2}$$

$$= 10 \times 5 + 5 \times 10 + 1 \times 10$$

$$= 110$$

2) Consider n-digit numbers where each digit is one of the 10 integers $0, 1, \ldots, 9$. How many such numbers are there for which no consecutive digits are equal?   (2 pts.)

$$\underbrace{10 \cdot 9 \cdot 9 \cdots 9}_{n-1} = 10 \cdot 9^{n-1}$$

3) Let A, B, and C be three events. Find expressions for the following   (1 pt.)
   a) Only A occurs

$$A \cap B^c \cap C^c$$

   b) At most one of the events occur   (1 pts.)

$$(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$$

   c) At most two of the events occur   (2 pts.)

At most two $\equiv$ complement of all 3

$$(A \cap B \cap C)^c .$$

4) Consider 3 urns. Urn A contains 2 white and 4 red balls, urn B contains 8 white and 4 red balls, and urn C contains 1 white and 3 red balls. If 1 ball is selected from each urn, what is the probability that the ball chosen from urn A was white given that exactly 2 white balls were selected? (4 pts.)

$B \equiv$ event 2 white balls are selected

$A \equiv$ event ball selected from A is white

$$P(A|B) = \frac{P(AB)}{P(B)}$$

$$B = \{(W, W, R), (W, R, W), (R, W, W)\}$$

$$P(B) = \frac{2}{6} \cdot \frac{8}{12} \cdot \frac{3}{4} + \frac{2}{6} \cdot \frac{4}{12} \cdot \frac{1}{4} + \frac{4}{6} \cdot \frac{8}{12} \cdot \frac{1}{4}$$

$$= \frac{1}{6} + \frac{1}{36} + \frac{1}{9} = \frac{11}{36}$$

$$AB = \{(W, W, R), (W, R, W)\} \quad ; \quad P(AB) = \frac{7}{36}$$

$$\Rightarrow P(A|B) = \frac{7}{11}$$

5) You ask your neighbor to water a sickly plant while you are on vacation. Without water it will die with probability 0.8; with water it will die with probability 0.15. You are 90 % sure that your neighbor will remember to water the plant.

a) What is the probability that the plant will be alive when you return? (2 pts.)

$P(\text{Die} | \text{w/o water}) = 0.8 \Rightarrow P(\text{Live} | \text{w/o water}) = 0.2$

$P(\text{Die} | \text{water}) = 0.15 \Rightarrow P(\text{Live} | \text{water}) = 0.85$

$P(\text{water}) = 0.90 \Rightarrow P(\text{No water}) = 0.1$

$P(\text{plant will be alive}) = (0.2)(0.1) + (0.85)(0.90)$

$$= 0.785$$

b) If the plant is dead upon your return, what is the probability that your neighbor forgot to water it? (3 pts.)

$P(\text{No water} | \text{plant is dead})$

$$= \frac{P(\text{No water } \& \text{ plant is dead})}{P(\text{plant is dead})}$$

$$= \frac{(0.80)(0.10)}{0.215}$$

$$= 0.3721$$

6) Consider two independent tosses of a fair coin. Let A be the event that the first toss results in heads, let B be the event that the second toss results in heads, and let C be the event that in both tosses the coin lands on the same side. Show that the events A, B, and C are pairwise independent, that is A and B are independent, A and C are independent, and B and C are independent, but the three are not independent. (5 pts.)

$$P(A) = \frac{1}{2}$$

$$P(B) = \frac{1}{2}$$

$$P(C) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$$

$$P(AB) = \frac{1}{4} = P(A) \cdot P(B) \quad \checkmark$$

$$P(AC) = P(C|A) \cdot P(A) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \cdot$$

$$= P(A) \, P(C) \quad \checkmark$$

$$P(BC) = P(C|B) \cdot P(B)$$

$$= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$= P(B) \cdot P(C) \quad \checkmark$$

$$P(ABC) = \frac{1}{4}$$

$$P(A) P(B) P(C) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

Hence A, B & C are not independent