King Fahd University of Petroleum & Minerals
Department of Math. & Stat.

Math 668 - Midterm Exam (132)
Time: 2 hours 00 mns

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Name: ID #
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Problem 1	/7
Problem 2	/12
Problem 3	/6
Problem 4	/15
Total	/40

Problem # 1. (7 marks) Let Ω be a bounded domain of $\mathbb{R}^n$ and $a < 0 < b$. Define

$$K = \left\{ v \in L^2(\Omega) \text{ s.t. } a \leq v(x) \leq b \text{ a.e. } x \text{ in } \Omega \right\}$$

a. Show that K is nonempty, closed, convex and bounded in $L^2(\Omega)$

b. Determine $u = \mathrm{P}_K f$, $f \in L^2(\Omega)$.

Problem # 2. (12 marks) Let Ω be a bounded domain of $\mathbb{R}^n, n \geq 3$. Let

$$K = \left\{ v \in H_0^1(\Omega) \text{ s.t. } |\nabla v(x)| \leq 1, \text{ a.e. } x \text{ in } \Omega \right\}$$

and define $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ by $A(u) = -\Delta u + |u|^{p-2}u$, $p > 2$.

a. Show that K is nonempty, closed, convex and bounded in $L^2(\Omega)$

b. Find a condition on p for which A is well defined.

c. Under this condition, show that, for each f in $L^2(\Omega)$,the problem

$$\int_\Omega \left[\nabla u.(\nabla v - \nabla u) + |u|^{p-2}u(v-u) \right] dx \geq \int_\Omega f(v-u), \ \forall v \in K$$

has a solution $u \in K$.

Problem # 3. (6 marks) Show that the operator $A : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ given by $A(u) = -\Delta u + 2u$ is self-adjoint.

Problem # 4. (15 marks) Let Ω be a bounded and smooth domain of $\mathbb{R}^n$. Consider the problem

$$(P) \qquad \begin{cases} u_t(x,t) - k\Delta u(x,t) = 0 & \text{in } \Omega \times (0,+\infty) \\ u(x,0) = u_0(x) & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} + \alpha u = 0 & \text{on } \partial\Omega \times [0,+\infty) \end{cases}$$

where k and α are positive constants and ν is the unit outer normal on Ω.

a. Show that $-\Delta : D(-\Delta) \to L^2(\Omega)$ is maximal monotone and self-adjoint, where

$$D(-\Delta) = \left\{ v \in H^2(\Omega) \text{ s.t. } \frac{\partial v}{\partial \nu} + \alpha v = 0 \text{ on } \partial\Omega \right\}$$

b. If $u_0 \in L^2(\Omega)$, show that (P) has a unique solution with a regularity to be specified.

Remark. From the trace theory, we have

$$\int_{\partial\Omega} v^2 \leq c \left[\int_{\Omega} v^2 + \int_{\partial\Omega} |\nabla v|^2 \right], \; \forall v \in H^1(\Omega)$$