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HW♯2 Math605.

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Exercise1:

Given that

$$\langle x^{2m}\rangle = \frac{\int_{-\infty}^{+\infty} x^{2m} e^{-\frac{1}{2}ax^2} dx}{\int_{-\infty}^{+\infty} e^{-\frac{1}{2}ax^2} dx}, \ m \in \mathbb{N},$$

show that

$$\langle x^{2m}\rangle = \frac{(2m-1)!!}{a^m},$$

where $(2m-1)!! = (2m-1)(2m-3)...5.3.1$

Exercise2(10 points)

Show that as $x \to \infty$

$$\int_x^\infty e^{-t} t^{\lambda-1}\, dt \sim x^\lambda e^{-x}\left[\frac{1}{x} + \frac{\lambda-1}{x} + \frac{(\lambda-1)(\lambda-2)}{x^2} + O\left(\frac{1}{x^4}\right)\right]$$

Solution: Use integration by parts

$$I = \int_x^\infty e^{-t} t^{\lambda-1}\, dt$$

$$\begin{aligned} u &= t^{\lambda-1} & dv &= e^{-t} \\ du &= (\lambda-1)t^{\lambda-1} & v &= -e^{-t} \end{aligned}$$

$$\begin{aligned}
I &= [-t^{\lambda-1}e^{-t}]_x^\infty + (\lambda-1)\int_x^\infty t^{\lambda-2e^{-t}}\, dt \\
&= x^{\lambda-1}e^{-x} + (\lambda-1)\left[[-t^{\lambda-1}e^{-t}]_x^\infty + \int_x^\infty (\lambda-2)t^{\lambda-3}e^{-t}\, dt\right] \\
& \qquad u = t^{\lambda-2} \qquad dv = e^{-t} \\
& \qquad du = (\lambda-2)t^{\lambda-3} \qquad v = -e^{-t} \\
&= x^{\lambda-1}e^{-x} + (\lambda-1)x^{\lambda-2}e^{-x} + (\lambda-1)(\lambda-2)\int_x^\infty e^{-t}t^{\lambda-3}\, dt \\
& \qquad u = t^{\lambda-3} \qquad dv = e^{-t} \\
& \qquad du = (\lambda-3)t^{\lambda-4} \qquad v = -e^{-t} \\
& \quad \vdots \\
&= x^{\lambda-1}e^{-x} + (\lambda-1)x^{\lambda-2}e^{-x} + (\lambda-1)(\lambda-2)(\lambda-3)x^{\lambda-3}e^{-x} \\
& \quad +(\lambda-1)(\lambda-2)(\lambda-3)\int_x^\infty t^{\lambda-4}e^{-t}\, dt \\
&= x^\lambda e^{-x}\left(\frac{1}{x} + \frac{(\lambda-1)}{x^2} + \frac{(\lambda-1)(\lambda-2)(\lambda-3)}{x^3} + \frac{R}{x^\lambda e^{-x}}\right)
\end{aligned}$$

$$
\begin{aligned}
R &= (\lambda-1)(\lambda-2)(\lambda-3)\int_x^\infty t^{\lambda-4}e^{-t} \\
&\leq |(\lambda-1)(\lambda-2)(\lambda-3)|\left|\int_x^\infty t^{\lambda-4}e^{-t}\,dt\right| \\
&\leq (\lambda-1)(\lambda-2)(\lambda-3)\int_x^\infty |t^{\lambda-4}|e^{-t}\,dt \\
& \quad x>0,\ \text{so} \\
&\leq (\lambda-1)(\lambda-2)(\lambda-3)\int_x^\infty t^{\lambda-4}e^{-t}\,dt \\
&= \frac{(\lambda-1)(\lambda-2)(\lambda-3)}{x^{4-\lambda}}\int_x^\infty t^{\lambda-4}x^{4-\lambda}e^{-t}\,dt \\
&= \frac{(\lambda-1)(\lambda-2)(\lambda-3)}{x^{4-\lambda}}\int_x^\infty \left(\frac{x}{t}\right)^{4-\lambda}e^{-t}\,dt \\
& \quad \text{But t} \geq \text{x. Therefore } \left(\frac{\text{x}}{\text{t}}\right) \leq 1 \\
\Rightarrow R &\leq \frac{(\lambda-1)(\lambda-2)(\lambda-3)}{x^{4-\lambda}}\int_x^\infty e^{-t}\,dt \\
&= \frac{(\lambda-1)(\lambda-2)(\lambda-3)}{x^{4-\lambda}}e^{-x} \\
& \quad \text{which, when divided by } x^{-1}e^{-x} \text{ is O}\left(\frac{1}{\text{x}^4}\right) \\
\text{Therefore I} &= x^\lambda e^{-x}\left(\frac{1}{X}+\frac{(\lambda-1)}{x}\right. \\
& \quad \left.+\frac{(\lambda-1)(\lambda-2)(\lambda-3)}{x^3}+O\left(\frac{1}{x^4}\right)\right)
\end{aligned}
$$

Implied constant $(\lambda-1)(\lambda-2)(\lambda-3)$.

Exercise3(10 points)

show that

$$\int_1^x \left(1+\frac{1}{t}\right)^t dt \sim xe - \frac{1}{2}e\log x + O(1)$$

Solution:

$\left(1+\frac{1}{t}\right)^t = e^{t\log(1+\frac{1}{t})}$

$\int_1^x \left(1+\frac{1}{t}\right)^t dt = \int_1^x e^{t\log(1+\frac{1}{t})}\,dt$

$$
\begin{aligned}
t\log\left(1+\frac{1}{t}\right) &= t\left[\frac{1}{t}-\frac{1}{2t^2}+\frac{1}{3t^3}-\cdots\right]
\end{aligned}
$$

Expand $\qquad \underline{\text{convergent}}\ \text{for}|t|>1(|\frac{1}{t}|<1)$

$$
= 1-\frac{1}{2t}+\frac{1}{3t^2}-\cdots
$$

Therefore

$$
\begin{aligned}
e^{t\log(1+\frac{1}{t})} &= e^{1-\frac{1}{2t}+\frac{1}{3t^2}-\cdots} \\
&= e\cdot e^{-\frac{1}{2t}+\frac{1}{3t^2}}-\cdots \\
&= e\left[1-\frac{1}{2t}+\frac{1}{3t^2}-\cdots+\frac{1}{2!}\left(-\frac{1}{2t}+\frac{1}{3t^2}+\cdots\right)^2+\cdots\right] \\
&= e\left[1-\frac{1}{2t}+O\left(\frac{1}{t^2}\right)\right] \text{ for |t|} > 1
\end{aligned}
$$

Thus

$$
\begin{aligned}
\int_1^x\left(1+\frac{1}{t}\right)^t dt &= e\int_1^x\left[1-\frac{1}{2t}+O\left(\frac{1}{t^2}\right)\right] dt \\
&= e\left[t-\frac{1}{2}\log t+O\left(\frac{1}{t}\right)\right]_1^x \\
&= e\left[x-\frac{1}{2}\log x+O(\frac{1}{x})\right] \text{ as x} \to +\infty \\
&\sim ex-\frac{1}{2}e\log x \text{ as x} \to +\infty \quad \text{as required.}
\end{aligned}
$$

Exercise4(10 points)

The modified Bessel function of the first kind may be defined by

$$
I_n(x)=\frac{1}{\pi}\int_0^\pi e^{x\cos t}\cos(nt)\,dt.
$$

Show that

$$
I_n(x)\sim\frac{e^x}{\sqrt{2\pi x}},\ x\to+\infty.
$$

Solution:

$I_n(x)=\dfrac{1}{\pi}\displaystyle\int_0^\pi e^{x\cos t}\cos(nt)\,dt$

On the interval $[0,\pi]$, the max. value of $\cos t$ (min. value of $-\cos t$) is at the endpoint $t=0$.

Therefore set

$u=h(t)-h(0)=-\cos t+1\ \ du=\sin t\,dt$

Thus

$I_n(x)=\dfrac{1}{\pi}\displaystyle\int_0^2 e^{-xu+x}\frac{\cos[(nt(u))]}{\sin[t(u)]}\,du$

$u\approx h''(0)\dfrac{(t-0)^2}{2}+\cdots\Rightarrow u\approx\dfrac{t^2}{2^2}$

$$
\begin{aligned}
I_n(x) &\approx \frac{e^x}{\pi}\int_0^2 e^{-xu}\frac{\cos(n\sqrt{2u})}{\sin(\sqrt{2u})}\,du \\
&\sim \frac{e^x}{\pi}\int_0^\infty e^{-xu}\frac{\cos(n\sqrt{2u})}{\sin(\sqrt{2u})}\,du \quad x\to+\infty \\
&\frac{\cos(n\sqrt{2u})}{\sin(\sqrt{2u})} \\
&\sim \frac{1-\frac{n^2 2u}{2}+\cdots}{\sqrt{2u}-\frac{2\sqrt{2}u^{\frac{3}{2}}}{6}+\cdots} \text{ as u} \to 0 \text{ and hence} \sim \frac{1}{\sqrt{2\mathrm{u}}} \\
&\sim \frac{e^x}{\pi}\int_0^\infty \frac{e^{-xu}}{\sqrt{2}\sqrt{u}}\,du \\
&\sim \frac{e^x}{\pi}\frac{\Gamma(+\frac{1}{2})}{\sqrt{2}\sqrt{x}} \\
&\sim \frac{e^x}{\sqrt{2\pi x}} \quad x\to+\infty
\end{aligned}
$$