King Fahd University of Petroleum and Minerals

Department of Mathematics and Statistics

Math 605
Final Exam– 2014–2015 (132)
Monday, May 19, 2014

Allowed Time: 150 minutes

Instructor: **Dr. Boubaker Smii**

Name: ________________________________

ID #: ____________________

Section #: ___________ Serial Number: ___________

Instructions:

1. Write clearly and legibly. You may lose points for messy work.
2. **Show all your work**. No points for answers without justification !

Question #	Grade	Maximum Points
1		21
2		19
3		20
4		16
5		15
6		29
Total:		120

Exercise 1:

Consider the modified Bessel function given by:

$$K_0(x) = \int_1^{\infty} (s^2 - 1)^{-\frac{1}{2}} \, e^{-xs} \, ds, \;\; x \geq 0.$$

a)- Show that $K_0(x) = e^{-x} \displaystyle\int_0^{\infty} (t^2 + 2t)^{-\frac{1}{2}} \, e^{-xt} \, dt.$

b)- For $\mid t \mid < 2$, Show that $\;(t^2 + 2t)^{-\frac{1}{2}} = (2t)^{-\frac{1}{2}} \displaystyle\sum_{n=0}^{\infty} (-\frac{t}{2})^n \frac{\Gamma(n + \frac{1}{2})}{n! \, \Gamma(\frac{1}{2})}$

Hint: You may use: $\begin{pmatrix} y \\ z \end{pmatrix} = \frac{\Gamma(y+1)}{\Gamma(z+1)\,\Gamma(y-z+1)}$

c)- Use Watson's Lemma to find an asymptotic expansion for $K_0(x)$, as $x \longrightarrow +\infty$.

Exercise 2:

I- Consider the integral

$$I(x) = \int_0^\infty \frac{e^{-t}}{1+xt}\, dt,\ \ x \geq 0.$$

For $x \geq 0$ and $N = 0, 1, 2, ...$ Prove that:

$$|I(x) - \left(1 - x + \cdots + (-1)^N\, N!\, x^N\right)| \leq (N+1)!\, x^{N+1}.$$

(**Justify clearly your answer !**)

II- Consider the series $\displaystyle\sum_{n=0}^{\infty} (-1)^n\, n!\,.$

a)- Test the given series for convergence or divergence.

b)- Find the Borel Sum of $\sum_{n=0}^{\infty} (-1)^n\, n!\,.$ (**Justify clearly your answer !**)

c)- If B is the Borel sum of $\displaystyle\sum_{n=0}^{\infty} (-1)^n\, n!$, then for $x \geq 0$ and $N = 0, 1, 2, ...$ Prove that:

$$|B - \left(1 - 1 + \cdots + (-1)^N\, N!\right)| \leq (N+1)!$$

Exercise 3:

Using the asymptotic matching method, find an approximate solution to the Boundary value problem:

$$\epsilon\, y'' + y' = \frac{1}{1+x^2}, \qquad y(0) = 0; \;\; \lim_{x \longrightarrow \infty} y(x) = 1.$$

Exercise 4:
Consider the initial value problem:

$$y'' = -\epsilon\, y' - 1; \quad y(0) = 0,\, y'(0) = 1. \tag{1}$$

Find an approximation solution for the initial value problem (1) to $O(\epsilon^2)$.

Exercise 5:

Consider the the Maclaurin expansion for the exponential function $f(z) = e^z$.

a)- Find the Padé approximation $P_1^1(z)$.

b)- Suppose that $z = 1$. Compare the result with the exact answer and the result obtained from the first three terms in the Macluarin series.

Exercise 6:

The Schrödinger equation is given by :

$$\epsilon^2 y'' - F(x)y = 0,\ F(x) \neq 0,\ \epsilon \longrightarrow 0^+. \tag{a}$$

1)- Use the WKB analysis to find an approximate solution to (a).

2)- Use part 1) to solve the Boundary value problem:

$$\begin{cases} \epsilon\, y'' = -y, \ \ \epsilon \longrightarrow 0^+. \\ y(0) = 0, \ \ y(1) = 1. \end{cases} \tag{b}$$