HMW #5

Prob 1:

Let (E, Σ, μ) be a measure space and $(f_n)_n$ a sequence of positive measurable functions converging simply to f. Assume that there exist K such that $\sup_{n \geq 0} \int_E f_n \, d\mu \leq K$.

(a) Prove that $\int_E f \, d\mu \leq K$. (b) Consider $([0,1], \mathcal{B}([0,1]), \lambda)$ and the seq. $(f_n)_n$ defined by $f_{2n} = \chi_{[0,1/2)}$ and $f_{2n+1} = \chi_{[1/2,1]}$. Compute $\int \limsup_n f_n \, d\lambda$ & $\limsup_n \int f_n \, d\lambda$.

Prob 2:

Compute the limit of the following sequences

(a) $\displaystyle\int_{\mathbb{R}} \chi_{\{x : 3|\cos \frac{x}{m}| \geq 2\}} \frac{e^{-x^2}}{2\cos\left(\frac{x}{m}\right) - 1} \, dx$, (b) $\displaystyle\sum_{m \geq 0} \frac{n}{m} \cdot \sin\left(\frac{1}{nm}\right)$.

Prob 3:

Let $f : (0,1) \to \mathbb{R}$ be a monotone positive function which is integrable. We define $g_n(x) := f(x^n)$. Find the $\displaystyle\lim_{n \to \infty} \int_{(0,1)} g_n \, d\lambda$.

Prob 4:

Find the limits of (i) $\displaystyle\int_0^1 \frac{n e^{-x} dx}{nx+1}$, (ii) $\displaystyle\int_{(0,\infty)} \frac{\sin x}{x^2} \frac{x^{\frac{1}{n}}}{1 + x^{\frac{1}{n}}} \, dx$

Prob 5:

Let the function f defined on $(0, +\infty)$ be defined by $f(t) = \displaystyle\int_0^\infty e^{-xt} \frac{1 - \cos x}{x} \, dx$. (a) Prove that f is differentiable for all $t \in (0, \infty)$ and compute explicitly its derivative (as a function of t only). (b) Find $\displaystyle\lim_{t \to +\infty} f(t)$.