HMW #4 MATH 531

Prob 1 Show that $\lim_{n\to\infty}\int_{\mathbb{R}^+} f_n \, d\mu$ converges and find its limit in the following four cases of $f_n : \mathbb{R}^+ \to \mathbb{R}$

(a) $f_n(x) = e^{-x} |\cos x|^{\frac{1}{n}}$

(b) $f_n(x) = \sin(nx) \cdot \chi_{[0,n]}(x)$

(c) $f_n(x) = \dfrac{n e^{-nx}}{\sqrt{1+n^2x^2}}$

(d) $f_n(x) = \dfrac{n e^{-x}}{\sqrt{1+n^2x^2}}$

Prob 2

Let (X, Σ, μ) be a measure space and $f : X \to \mathbb{R}$ a positive measurable function. For all $t > 0$ we define

$M_f(t) = \{x \in X : f(x) > t\}$ and $\varphi_f(t) = \mu(M_f(t))$.

Show that $\int_X f \, d\mu = \int_0^{\infty} \varphi_f(t) \, dt$.

Prob 3

Let (X, Σ, μ) be a measure space and $(C_n)_{n\geq 1}$ a sequence of measurable sets. Let $f : (X, \Sigma) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ an integrable function s.t. $\int_X |f - \chi_{C_n}| \, d\mu \xrightarrow[n\to\infty]{} 0$

(a) Show that $|f| \leq 2$ μ-a.e.

(b) Show that $\exists\, C \in \Sigma$ s.t. $f = \chi_C$ μ-a.e

(c) Prove that if $\sum_{n\geq 0} \mu(C_n \Delta C) < +\infty$ then $\chi_{C_n} \to \chi_C$ μ-a.e.