HMW #3

(I) Let $\mu^*: \mathcal{B}(\mathbb{R}) \to [0,+\infty]$ be defined by
$\mu^*(E) = \sup\{ d-c ; (c,d) \subset E \}$.
Is this function monotone? Is is invariant by translation? Does it preserve the length of the intervals? Is it sub-additive?

(II) Verify the following relations

$$\chi_{E\cap F} = \chi_E \cdot \chi_F \; ; \; \chi_{E+F} = \chi_E + \chi_F$$

$$\chi_{E\cup F} = \chi_E + \chi_F - \chi_{E\cap F} \; ; \; \chi_{\bigcup_n E_n} = \sup \chi_{E_n}$$

(III) Prove that $\chi_{\mathbb{Q}}(t)$ is discontinuous everywhere.

(IV) Let $\{F_n\}_{n\geq 1}^{\infty}$ be a sequence of measurable subsets of $\mathbb{R}^N$ and that $\sum_{n\geq 1}^{\infty} m(F_n) < \infty$
Define $F = \bigcap_{n\geq 1}^{\infty} \bigcup_{k\geq n} E_k$. Show that $m(F)=0$.

(V) Let A be a subset of $\mathbb{R}$ s.t. $m(A)=0$.
Show that the set $B = \{x^2 ; x \in A\}$ also has measure zero.

(VI) Let $f: \mathbb{R} \to \mathbb{R}$ be a monotone function.
Prove that f is Borel measurable.