FINAL MATH 531

(I) Let $F(u)=\int^{+\infty} \frac{\sin t}{t} e^{-ut}\,dt$, $u>0$. Find the value of $F(u)$ (in terms of u). Justify all your steps.

(II) Compute the integrals: (a) $\lim_{n\to+\infty} \int_{-\infty}^{+\infty} \frac{n}{x(1+x^2)} \sin\frac{x}{n}\,dx$
(b) $\lim_{n\to+\infty} \int_0^{+\infty} e^{-x} \arctan\frac{x}{n}\,dx$

(III) (a) Prove that $\forall z\geq 0$, $0\leq 1-e^{-z}\leq z$
(b) Deduce that $\forall y>0$, $x\mapsto \frac{1-e^{-x^2y}}{x^2}$ is integrable on $[0,\infty)$
(c) For $y>0$, let $F(y)=\int_0^{+\infty} \frac{1-e^{-x^2y}}{x^2}\,dx$, Prove that F is differentiable on $(0,+\infty)$. Compute $F'(y)$. We recall that $\int_0^{+\infty} e^{-x^2}dx=\sqrt{\pi}/2$. (d) Deduce $F(y)$ modulo a constant. (e) Compute this constant by looking at $\lim_{n\to+\infty} F\left(\frac{1}{n}\right)$.

(IV) Let (E,Σ,μ) be a measure space, $f: E\to[0,+\infty)$ a positive measurable fct such that $0<\int_E f\,d\mu<+\infty$. Find, in terms of $\alpha\in\mathbb{R}_+^*$; $\lim_{n\to+\infty}\int_E n\ln\left(1+\left(\frac{f(x)}{n}\right)^\alpha\right)d\mu(x)$.
(Hint: $1+t^\alpha\leq(1+t)^\alpha$, $t\geq 0$, $\alpha>1$.)

(V) Let $f(x)=\int_0^{+\infty} e^{-t^2}\cos(tx)\,dt$, $x\in\mathbb{R}$. (1) Prove that $f\in C^1(\mathbb{R})$, (2) Prove that f satisfies $y'=-\frac{x}{2}y$.
(3) Deduce an explicit expression of f.