KFUPM, DEPARTMENT OF MATHEMATICS AND STATISTICS

MATH 521: FINAL EXAM, TERM (132), MAY 20, 2014

Time: 07:00(pm) to 10:00

Name : ....

ID : .... Section : ....

Exercise 1. Let X be a topological space and $A \subseteq X$.

(1) Show that $X - \overline{A} = \text{int}(X - A)$.

(2) Show that $\text{Fr}(A) = \text{Fr}(X - A)$.

(3) Show that $\text{int}(\overline{A} - A) = \emptyset$.

(4) Show that if A is closed or open, then $\text{int}(\text{Fr}(A)) = \emptyset$.

Exercise 2. An open set O in a topological space is said to be regular open if O is the interior of its closure. A closed set is called regularly closed if it is the closure of its interior.

(1) Show that the complement of a regularly open set is regularly closed and; vice versa.

(2) Give an open sets in $\mathbb{R}$ which is not regularly open.

(3) Show that if A is any subset of a topological space, then $\operatorname{int}(\overline{A})$ is regularly open.

(4) Show that the intersection of two regularly open sets is regularly open. Give an example of two regularly open sets such that the union is not regularly open.

Exercise 3. Let $(X, \leq)$ be a totally ordered set. Recall that the order topology on X is the topology with basis

$$\mathcal{B} = \{]a, b[: a, b \in X\} \bigcup \{] \leftarrow, a[: a \in X\} \bigcup \{]a, \rightarrow [: a \in X\} .$$

(1) Show that X is connected if and only if the following properties hold.
  (i) Every nonempty set with an upper bound has a supremum.
  (ii) For all $x < y$ in X, the interval $]x, y[$ is nonempty.

(2) Show that if X is connected, then the connected subsets of X are the intervals.

Exercise 4. Let $f : X \longrightarrow Y$ be a continuous map.

(1) Show that if X is path connected, then the graph $G(f) = \{(x, f(x)) : x \in X\}$ is path connected in $X \times Y$.

(2) Give an example of continuous function with non path connected graph.

(3) Let $A = \{(x, \sin(\frac{1}{x})) : 0 < x \leq 1\}$. Show that A is path connected but $\overline{A}$ is not path connected.

Exercise 5. Let (X,d) be a compact metric space. For $\varepsilon > 0$ and $x, y \in X$, by an ε-chain joining x and y, we mean a finite sequence $x_0, x_1, \dots, x_n$ such that $x_0 = x, x_n = y$ and $d(x_i, x_{i+1}) \leq \varepsilon$, for every $i \in \{0, 1, \dots, n-1\}$. Show that the following statements are equivalent:

(i) X is connected,

(ii) for each $\varepsilon > 0$ and all $x, y \in X$, there exists an ε-chain joining x and y.

Exercise 6. Let X be a topological space and R be an equivalence relation on X. Let $\pi : X \longrightarrow X/R$ be the canonical surjection and $G = \{(x, y) \in X \times X : \pi(x) = \pi(y)\}$

(1) Suppose that X is a compact Haussdorff space. Show that the following statements are equivalent.
   - (i) X/R is Haussdorff;
   - (ii) G is closed;
   - (iii) π is a closed map.

(2) Show that if X is a normal space and π is a closed map, then X/R is normal.

Exercise 7. Show that the function $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{R}$ defined by

$$f(x, y) = (\frac{\sin(x + y)}{2} + 7, \frac{\cos(x + y)}{2} - 11)$$

has a unique fixed point.