Math 513-132 Quiz 5

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Q.1: Find the steady–state temperature in the circular cylinder of radius 2 and height 4 by solving $\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}=0$, with $u(r,0)=5$, $u(r,4)=0$ and the lateral side is insulated.

Let $u(r,z)=0$, then we get

$Z''-\lambda Z=0$

$rR''+R'+r\lambda R=0 \rightarrow \circledast$

$\left.\frac{\partial u}{\partial r}\right|_{r=2}=0 \Rightarrow R'(2)=0$

The solution of $\circledast$ is

$R(r)=c_1J_0(\alpha r)+c_2Y_0(\alpha r)$

$Y_0(\alpha r)\rightarrow-\infty$ as $r\rightarrow 0^+ \Rightarrow c_2=0$

$R'(r)=c_1\alpha J_0'(\alpha r)$

$R'(2)=0 \Rightarrow J_0'(2\alpha)=0$

Since $J_0'(2\alpha)=-J_1(2\alpha)$

and $J_1(2\alpha)=0 \rightarrow (**)$

$\Rightarrow \alpha_1=0$ and $\alpha_i>0$ are the solutions. (see fig 5.3)

And $J_0(0)=1$, So

$R_0(r)=c_1J_0(0)=c_1$

Also $Z''=0 \Rightarrow Z(z)=c_3+c_4z$

$Z(4)=0 \Rightarrow c_3=-4c_4$

$Z(z)=c_4(z-4)$

For $\alpha_i>0$, $i=2,3,\dots$

$R(r)=c_1J_0(\alpha_i r)$

and $Z''-\alpha_i^2Z=0$

$\Rightarrow Z(z)=c_5\cosh\alpha_iz+c_6\sinh\alpha_iz$

$Z(4)=0 \Rightarrow c_5=-c_6\frac{\sinh\alpha_i4}{\cosh\alpha_i4}$

$Z(z)=c_6\frac{\sinh(z-4)\alpha_i}{\cosh 4\alpha_i}$

$u(r,z)=A_1(z-4)+\sum_{i=2}^{\infty}A_i\frac{\sinh(z-4)\alpha_i}{\cosh 4\alpha_i}J_0(\alpha_ir)$

$u(r,0)=5$

$\Rightarrow 5=-4A_1+\sum_{i=2}^{\infty}\frac{-A_i\sinh 4\alpha_i}{\cosh 4\alpha_i}J_0(\alpha_ir)$

$-4A_1=\frac{2}{4}\int_0^2 5r\,dr=\frac{5}{2}\cdot\frac{r^2}{2}=5$

$A_1=-\frac{5}{4}$

$-A_i\frac{\sinh 4\alpha_i}{\cosh 4\alpha_i}=\frac{2}{4J_0^2(2\alpha_i)}\int_0^2 5rJ_0(\alpha_ir)\,dr$

$=\frac{5J_1(2\alpha_i)}{\alpha_iJ_0^2(2\alpha_i)}=0$

From $(**)$.

$u(r,z)=\frac{5}{4}(4-z)$.

Q.2: Solve $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} = \dfrac{\partial^2 u}{\partial t^2}$ to find the displacement $u(r,t)$ in a circular membrane of radius 4 clamped along its circumference if its initial displacement is 0 and it has initial unit velocity in upward direction.

$0 < r < 4$, $u(r,0) = 0$

$u(4,t) = 0$ $\quad u_t(r,0) = 1$

Let $u(r,t) = R(r)\,T(t)$

Then $T'' + \lambda T = 0$

and $rR'' + R' + \lambda r R = 0$

For $\lambda = \alpha^2 > 0$, $\alpha > 0$,

$R(r) = c_1 J_0(\alpha r) + c_2 Y_0(\alpha r)$

$Y_0(\alpha r) \to -\infty$ as $r \to 0^+$

$\Rightarrow \quad c_2 = 0$

$u(4,t) = 0 \Rightarrow R(4) = 0$

$\Rightarrow \quad J_0(4\alpha) = 0$

Let α_i, $i = 1, 2, \ldots$ be the solutions of this equation.

$\lambda_i = \alpha_i^2$ are the eigenvalues.

$T'' + \alpha_i^2 T = 0$

$T(t) = c_3 \cos\alpha_i t + c_4 \sin\alpha_i t$

$u(r,0) = 0 \Rightarrow T(0) = 0$

$\Rightarrow \quad c_3 = 0$

$$u(r,t) = \sum_{i=1}^{\infty} A_i \sin\alpha_i t \, J_0(\alpha_i r)$$

$$u_t(r,t) = \sum_{i=1}^{\infty} A_i \alpha_i \cos\alpha_i t \, J_0(\alpha_i r)$$

$u_t(r,0) = 1$

$$\Rightarrow 1 = \sum_{i=1}^{\infty} \alpha_i A_i \, J_0(\alpha_i r)$$

$$\alpha_i A_i = \frac{2}{16\, J_1^2(4\alpha_i)} \int_0^4 r\, J_0(\alpha_i r)\, dr$$

$$A_i = \frac{1}{8\alpha_i\, J_1^2(4\alpha_i)} \cdot \frac{1}{\alpha_i^2} \cdot 4\alpha_i J_1(4\alpha_i)$$

$$= \frac{1}{2\alpha_i^2\, J_1(4\alpha_i)}$$

$$u(r,t) = \sum_{i=1}^{\infty} \frac{\sin\alpha_i t \; J_0(\alpha_i r)}{2\alpha_i^2\, J_1(4\alpha_i)}$$