Math ~~301~~ 513-132 Quiz ~~5~~ 4

Name:..Sec#:...............ID#:............................Ser#:.........

Q.1: Find the displacement $u(x,t)$ in a string of length 1 if the string is clamped at the ends $x=0$, $x=1$ and starts from rest with initial position $3\sin(2\pi x)+4\sin(5\pi x)$.

Separation of Variables

Let $u(x,t) = X(x)T(t)$, then

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad 0<x<1,\ t>0$$

$$u(0,t)=0, \quad u(1,t)=0$$

$$u(x,0) = 3\sin 2\pi x + 4\sin 5\pi x$$

$$u_t(x,0)=0$$

$$\Rightarrow \frac{X''}{X} = \frac{T''}{T} = -\lambda$$

$$X''+\lambda X=0, \quad X(0)=0,\ X(1)=0$$

$\lambda=0$ Trivial sol.

$\lambda=-\alpha^2$ Trivial Sol.

$\lambda=\alpha^2, \quad \alpha>0$

$$X(x) = A\cos\alpha x + B\sin\alpha x$$

$$X(0)=0 \Rightarrow A=0$$

$$X(1)=0 \Rightarrow B\sin\alpha=0$$

Let $B\neq 0$, $\Rightarrow \alpha = n\pi$

$$\lambda = n^2\pi^2$$

$$X_n(x) = B\sin n\pi x$$

$$T''+n^2\pi^2 T=0$$

$$T'(0)=0$$

$$T(t) = C\cos n\pi t + D\sin n\pi t$$

$$T'(t) = -Cn\pi\sin n\pi t + Dn\pi\cos n\pi t$$

$$T'(0)=0 \Rightarrow D=0$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin n\pi x \cos n\pi t$$

$$u(x,0) = 3\sin 2\pi x + 4\sin 5\pi x$$

$$\Rightarrow 3\sin 2\pi x + 4\sin 5\pi x = \sum_{n=1}^{\infty} A_n \sin n\pi x$$

$\Rightarrow A_2 = 3, \quad A_5 = 4$

$A_n = 0$ for all $n\neq 3,4$

$$u(x,t) = 3\sin 2\pi x\cos 2\pi t + 4\sin 5\pi x\cos 5\pi t$$

Q.2: Find the displacement $u(x,t)$ in a string of length 1 if the string is clamped at the ends $x = 0$, $x = 1$, with initial position $u(x,0) = 0$ and with initial velocity $u_t(x,0) = 3\sin(2\pi x) + 4\sin(5\pi x)$.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

$u(0,t) = 0$, $u(1,t) = 0$

$u(x,0) = 0$

$u_t(x,0) = 3\sin 2\pi x + 4\sin 5\pi x$

Taking Laplace transform

$$\frac{d^2U}{dx^2} = s^2U - su(x,0) - u_t(x,0)$$

$$\frac{d^2U}{dx^2} - s^2U = -3\sin 2\pi x - 4\sin 5\pi x$$

$$U_c = A\cosh sx + B\sinh sx$$

Let $U_p = C\sin 2\pi x + D\sin 5\pi x$

Then

$$U_p'' = -4\pi^2 C\sin 2\pi x - 25\pi^2 D\sin 5\pi x$$

$$-4\pi^2 C\sin 2\pi x - 25\pi^2 D\sin 5\pi x$$
$$- s^2 C\sin 2\pi x - s^2 D\sin 5\pi x$$
$$= -3\sin 2\pi x - 4\sin 5\pi x$$

$$\Rightarrow (s^2+4\pi^2)C = 3 \qquad C = \frac{3}{s^2+4\pi^2}$$

$$D = \frac{4}{s^2+25\pi^2}$$

$$U(x,s) = A\cosh sx + B\sinh sx + \frac{3}{s^2+4\pi^2}\sin 2\pi x + \frac{4}{s^2+25\pi^2}\sin 5\pi x$$

$u(0,t) = 0 \Rightarrow U(0,s) = 0 \Rightarrow A = 0$

$u(1,t) = 0 \Rightarrow U(1,s) = 0$

$\Rightarrow B\sinh s = 0$

$\Rightarrow B = 0$

$$U(x,s) = \frac{3}{s^2+4\pi^2}\sin 2\pi x + \frac{4}{s^2+25\pi^2}\sin 5\pi x$$

$$u(x,t) = \frac{3}{2\pi}\sin 2\pi t\,\sin 2\pi x + \frac{4}{5\pi}\sin 5\pi t\,\sin 5\pi x$$