King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
MATH430 - Introduction to Complex Variables
Final Exam – Term 132 (2013–2014)

Exercise 1

Determine all the isolated singularities of the function $\dfrac{z-1}{\cos z}$ and compute the residue at each singularity.

Exercise 2
Using the method of the residue, show that

$$\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} = \frac{2\pi}{\sqrt{1-a^2}}, \quad |a| < 1.$$

Exercise 3
Compute

$$\text{p.v.} \int_{-\infty}^{+\infty} \frac{\cos(\alpha x)}{x - w} \, dx$$

where $\alpha \in \mathbb{R} \setminus \{0\}$ and $w \in \mathbb{C}$.

Exercise 4

1) Consider the function

$$f(z) = \frac{1}{q^2(z)}$$

where q is analytic at z_0, $q(z_0) = 0$ and $q'(z_0) \neq 0$. Show that z_0 is a pole of order 2 of the function f, with

$$\operatorname{Res}(f, z_0) = -\frac{q''(z_0)}{[q'(z_0)]^3}.$$

2) Application: Find the residue at $z = 0$ of the functions

(a) $f(z) = \csc^2 z$

(b) $f(z) = \dfrac{1}{(z+z^2)^2}$

Exercise 5

1. Below is an outline of a proof of the fact that *for any analytic function f that is never zero in a simply connected domain D there exists a single-valued branch of* $\log f$ *analytic in D*. Justify each step in the proof

(a) $\dfrac{f'}{f}$ has an anti-derivative in D, say H.

(b) The function $f(z)e^{-H(z)}$ is constant in D, so that $f(z) = ce^{H(z)}$.

(c) Letting α be a value of $\log c$, the function $H(z) + \alpha$ is a branch of $\log f(z)$ analytic in D.

2. Application: Prove that there exists an analytic g in the unit disk $|z| < 1$ such that $g^2(z) = z^n - 1$, where $n \geq 1$.

Exercise 6

(a) Prove that if f has a pole of order m at z_0, then f' has a pole or order $m+1$ at z_0.

(b) Prove that if f has a pole of order m at z_0, then $g(z) = \frac{f'}{f}$ has a simple pole at z_0. Find $\text{Res}(g, z_0)$.

(c) Let f have an isolated singularity at z_0. Show that $\text{Res}(f', z_0) = 0$.

Exercise 7 Suppose that f is analytic on and *outside* the simple closed *negatively* oriented contour Γ. Assume further that f is analytic at ∞ and $f(\infty) = 0$. Prove that

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(w)}{w - z} dw$$

for all z outside Γ. [HINT: Apply Cauchy's integral formula for z in an annulus and let the outer radius tend to ∞]

Exercise 8

(a) Find all the functions that are analytic everywhere in the extended complex plane.

(b) Classify all the functions that are analytic in the extended complex plane except for a pole at one point.