KFUPM - Department of Mathematics and Statistics
MATH 345, Term 132
Final Exam (Out of 100), Duration: 180 minutes

NAME:
ID:

Solve the following Exercises.

Exercise 1 (15 points 5-5-5): Let G be a cyclic group.
(1) If 6 divides $|G|$, How many elements of order 6 does G have? Justify.
(2) If 8 divides $|G|$, How many elements of order 8 does G have? Justify.
(3) If a is an element of G with $|a| = 8$. Find all other elements of order 8 of G. Justify.

Exercise 2 (10 points 5-5): Let G be a finite abelian group with $|G| = pq$ where p and q are distinct positive prime integers. Set $H_1 = \{x \in G | x^q = e\}$ and $H_2 = \{x \in G | x^p = e\}$.
(1) Prove that H_1 and H_2 are incomparable cyclic groups and find their orders.
(2) Prove that G is the internal direct product of H_1 and H_2.

Exercise 3 (15 points 5-5-5): Let A and B be commutative rings, $Z(A)$ and $Z(B)$ be the sets of all zero-divisors of A and B respectively, and $\phi : A \longrightarrow B$ a ring isomorphism.
(1) Prove that $\phi(Z(A)) = Z(B)$.
(2) Find $Z(\mathbb{Z}_4)$ and $Z(\mathbb{Z}_2 \bigoplus \mathbb{Z}_2)$.
(3) Use (1) to prove that $\mathbb{Z}_4$ is not isomorphic to $\mathbb{Z}_2 \bigoplus \mathbb{Z}_2$. (No other justification is accepted).

Exercise 4 (20 points, 5-5-5-5): Let D be an integral domain and K a field.
(1) Prove that $K[X]$ is a PID (Principal Ideal Domain).
(2) Prove that $D[X]$ is a PID if and only if D is a field.
(3) Let $I = \{f(X) \in K[X] | f(a) = 0 \text{ for all } a \in K\}$. Prove that I is an ideal of $K[X]$.
(4) Assume that K is finite. Find a monic polynomial $g \in K[X]$ such that $I = (g)$.

Exercise 5 (20 points, 5-5-5-5): Let $f = a + bX + X^2$ be a polynomial in $\mathbb{Z}[X]$.
(1) Prove that f is reducible over $\mathbb{Z}$ if and only if $f(x)$ has a root in $\mathbb{Z}$.
(2) Let p be a positive prime integer and set $g = a + bX + pX^2$ where p do not divide ab. Prove that g is reducible over $\mathbb{Z}$ only if g has a root in $\mathbb{Z}$.
(3) Application: Use (2) to prove that $2X^2 + X + 1$ is irreducible over $\mathbb{Z}$.
(4) Is the Mod p Test applicable? Justify.

Exercise 6 (20 points, 5-5-5-5): Let $D = \mathbb{Z}[\sqrt{-5}]$.
(1) Prove that $1 + \sqrt{-5}$ and $1 - \sqrt{-5}$ are irreducible but not prime.
(2) Prove that 2 and 3 are irreducible in D.
(3) Find two factorizations of 6 in D.
(4) Is D a UFD (Unique Factorization Domain)? Justify.