KFUPM - Department of Mathematics and Statistics
MATH 345, Term 132
Exam III (Out of 80), Duration: 120 minutes

NAME:
ID:

Solve the following Exercises.

Exercise 1 (15 points 5-5-5): Let R and S be subrings of a ring T.
(1) Prove that $R \cap S$ is a subring of T.
(2) Under which condition $R \cup S$ is a subring of T?
(3) Find an example of a ring T with two subrings R and S such that $R \cup S$ is not a subring of T.

Exercise 2 (15 points 5-5-5): Let R and T be commutative rings, $f : R \longrightarrow T$ be a ring homomorphism and I an ideal of R.
(1) Prove that if f is onto then $f(I)$ is an ideal of T.
(2) Prove that if J is an ideal of T, then $E = \{x \in R | f(x) \in J\}$ is an ideal of R.
(3) Find an example of commutative rings R and T, $f : R \longrightarrow T$ a ring homomorphism and I an ideal of R such that $f(I)$ is not an ideal of T. (Hint you may use $\mathbb{Z}, \mathbb{R}$ and any ideal of $\mathbb{Z}$).

Exercise 3 (10 points 5-5): Let R be a commutative ring with unity.
(1) Prove that if R is of characteristic zero, then $\mathbb{Z}$ is isomorphic to a subring of R.
(2) Prove that if R is of prime characteristic p, then $\mathbb{Z}/p\mathbb{Z}$ is isomorphic to a subring of R.

Exercise 4 (10 points 5-5): Let R be a commutative ring and M and N two distinct maximal ideals of R.
(1) Prove that $M + N = R$, and $M \cap N = MN$.
(2) Give an example of a commutative ring with two maximal ideals M and N such that $M + N = R$.

Exercise 5 (15 points, 5-5-5):
(1) Find all ring automorphisms of $\mathbb{Z}$.
(2) Find all ring automorphisms of $\mathbb{Q}$.
(3) Is there any ring isomorphism from $\mathbb{Z}$ to $\mathbb{Q}$, justify?.

Exercise 6 (15 points, 5-5-5):
(1) Let m and n be two different positive integers. Prove that there is no isomorphism between $m\mathbb{Z}$ and $n\mathbb{Z}$.
(2) Prove that the quotient field K of an integral domain R is the intersection of all fields containing R.
(3) What is the field of fractions of the ring $R = \mathbb{Z}[i] = \{a + bi | a, b \in \mathbb{Z}\}$.