KFUPM - Department of Mathematics and Statistics
MATH 345, Term 132
Exam II (Out of 80), Duration: 120 minutes

NAME:
ID:

Solve the following Exercises.

Exercise 1 (10 points 5-5):
Let p be a (fixed) nonzero real number and $(\mathbb{R}^*, \times)$ be the multiplicative group of all nonzero real numbers under the usual multiplication. We define a new operation $\bigotimes$ on $\mathbb{R}^*$ by $x \bigotimes y = \frac{1}{p}xy$.
(1) Prove that $(\mathbb{R}^*, \bigotimes)$ is a group.
(2) Prove that $(\mathbb{R}^*, \times)$ is isomorphic to $(\mathbb{R}^*, \bigotimes)$.

Exercise 2 (15 points 5-5-5):
Let $\mathbb{Z}[\sqrt{2}]$ be the additive group of all elements of the form $\{x+y\sqrt{2}|x, y \text{ integers }\}$. For a fixed integer a, we define the following two maps φ_a and ψ_a :$\mathbb{Z}[\sqrt{2}] \longrightarrow \mathbb{Z}[\sqrt{2}]$ by $\varphi_a(x+y\sqrt{2}) = x + ay + y\sqrt{2}$ and $\psi_a(x+y\sqrt{2}) = x + ay - y\sqrt{2}$.
(1) Prove that φ_a and ψ_a are automorphisms of the additive group $(\mathbb{Z}[\sqrt{2}], +)$ with $\varphi(1) = \psi(1) = 1$.
(2) Use (1) to find all automorphisms ϕ of $\mathbb{Z}[\sqrt{2}]$ satisfying $\phi(1) = 1$.
(3) Find two infinite groups $H \subsetneqq G$ such that $Aut(H)$ is finite and $Aut(G)$ is infinite.

Exercise 3 (15 points 5-5-5): Let G and G' be groups, H a normal subgroup of G and $\phi : G \longrightarrow G'$ a group homomorphism.
(1) Prove that if ϕ is onto, then $\phi(H)$ is a normal subgroup of G'.
(2) Let $G = \mathbb{R}^*$ be the multiplicative group of nonzero real numbers, $H = \mathbb{Q}^*$ its normal subgroup of nonzero rational numbers, $G' = GL_2(\mathbb{R}) = \{$ all 2×2 matrices A with $detA \neq 0\}$ and $\phi : G \longrightarrow G'$ defined by $\phi(a) = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$.
Prove that ϕ is a group homomorphism which is not onto.
(3) Prove that $\phi(\mathbb{Q}^*)$ is not normal in G'. (Hint use the matrix $\begin{pmatrix} 1 & 0 \\ 1 & \sqrt{2} \end{pmatrix}$ and $a = 3$)

Exercise 4 (15 points 5-5-5): Let m and n be two positive integers that are relatively prime and $\phi : \mathbb{Z}/mn\mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z} \bigoplus \mathbb{Z}/n\mathbb{Z}$ defined by $\phi(\overline{x}[mn]) = (\overline{x}[m], \overline{x}[n])$.
(1) Prove that ϕ is an isomorphism.
(2) Let G_1, G_2 and H be three groups (denoted multiplicatively). Prove that if G_1 is isomorphic to H, then $G_1 \bigoplus G_2$ is isomorphic to $H \bigoplus G_2$.
(3) Use (1) and (2) to find at least an isomorphism between $\mathbb{Z}/6\mathbb{Z} \bigoplus \mathbb{Z}/5\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \bigoplus \mathbb{Z}/3\mathbb{Z} \bigoplus \mathbb{Z}/5\mathbb{Z}$.

Exercise 5 (10 points, 5-5):
(1) Find all subgroups of index 2 of the multiplicative group $\mathbb{R}^*$.
(2) Prove that the additive group $\mathbb{Q}$ has no subgroup of finite index.

Exercise 6 (15 points, 4-7-4):
Let $G = \mathcal{M}_n(\mathbb{R})$ be the additive group of all $n \times n$ matrices, H_1 the set of all $n \times n$ symmetric matrices (i. e. $A = A^T$) and H_2 be the set of all $n \times n$ skew symmetric matrices (i. e. $A = -A^T$).
(1) Prove that H_1 and H_2 are subgroups of G.
(2) Prove that G is the internal direct Product of H_1 and H_2.
(3) Are H_1 and H_2 isomorphic?