KFUPM - Department of Mathematics and Statistics
MATH 345, Term 132
Exam I (Out of 80), Duration: 120 minutes

NAME:
ID:

Solve the following Exercises.

Exercise 1 (10 points: 5-5): Let G be the multiplicative group of all 2×2 matrices with determinant equal to 1, and let $A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.
1-Find the centralizer $\mathcal{C}(A)$ of A.
2-Prove that for any $B, C \in \mathcal{C}(A)$, $BC = CB$.

Exercise 2 (10 points 5-5): Let G be an abelian group containing three distinct elements a, b, and c of order 2 and such that $ab \neq c$.
(1) Prove that G contains a subgroup H of order 8.
(2) Find an abelian group with three distinct elements of order 2 which has no subgroup of order 8.

Exercise 3 (20 points 5-5-5-5): (1) Let G be a multiplicative monoid (equivalently G has an operation which is associative and which has an identity element). If for every $a \in G$, the set $\{a^n | n \geq 0\}$ is finite, is G a group? Justify.
(2) Let G be a group, $\{H_n\}_{n\geq 1}$ be a chain of subgroups of G (i.e. $H_n \subseteq H_{n+1}$ for each n), and $H = \bigcup_{n\geq 1} H_n$. Prove that H is a subgroup of G.
(3) Let H and H' be subgroups of a group G. Prove that $H \cup H'$ is a subgroup of G if and only if H and H' are comparable for the inclusion, (equivalently $H \subseteq H'$ or $H' \subseteq H$).
4-Give an example of a group G with two subgroups H and H' such that $H \cup H'$ is not a subgroup of G. Justify.

Exercise 4 (15 points 5-5-5): Let G be an abelian finite group such that $|G| = 10$.
(1) Is G a cyclic group? Justify.
(2) If G is an abelian group with $G = pq$ where $p \neq q$ are prime integers, is G a cyclic group?
(3) If $|G| = p^2$ for some positive prime integer p, does G contain an element of order p^2? Is G cyclic? Justify.

Exercise 5 (10 points):
Let G be an abelian finite group which has exactly one nontrivial proper subgroup.
(1) Prove that G is cyclic.
(2) Prove that $|G| = p^2$ for some positive prime integer p.

Exercise 6 (15 points 5-5-5):
(1) Let G be a group of permutations of a set A and let $a \in A$. Set $stab(a) = \{\sigma \in G | \sigma(a) = a\}$. Prove that $stab(a)$ is a subgroup of G.
In the symmetric group S_4, find:
(2) A cyclic group of order 4.
(3) A non-cyclic group of order 4.