King Fahd University of Petroleum and Minerals

Department of Mathematics and Statistics

MATH 302 – Final Exam (Term 132)

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Duration: **3** Hours

Name : Solution **ID #** :______________________

Section # : ___________

Q1. Let $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

(a) Find a matrix **P** that diagonalizes **A** and the diagonal matrix **D** such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$

(b) Show that

$$\mathbf{A}^{10} = \frac{1}{2}\begin{pmatrix} 3^{10}+1 & 3^{10}-1 \\ 3^{10}-1 & 3^{10}+1 \end{pmatrix}$$

(Hint: *Use the fact that if* m *is a positive integer, then* $\mathbf{A}^m = \mathbf{P}\mathbf{D}^m\mathbf{P}^{-1}$)

Solution:

(a) The characteristic equation of A is $|A - \lambda I| = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = 0.$

This gives the eigenvalues of A: $\lambda_1 = 1$ and $\lambda_1 = 3$.

The corresponding eigenvectors are

$$K_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \ and \ K_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus, a matrix P that diagonalizes A is given by

$$P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

and the diagonal matrix D is

$$D = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}.$$

(b) We have

$$A^{10} = PD^{10}P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^{10}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3^{10} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3^{10}+1 & 3^{10}-1 \\ 3^{10}-1 & 3^{10}+1 \end{pmatrix}.$$

Q2. Let C be the line segment from $(1,2)$ to $(-2,-1)$. Evaluate

$$\int_C (x^2 + y)\, ds.$$

Solution:

Parametric equations for the line segment are

$$y = x + 1, \quad 1 \geq x \geq -2.$$

We obtain

$$ds = \sqrt{1 + [y'(x)]^2}\, dx = \sqrt{2}\, dx.$$

Thus,

$$\int_C (x^2 + y)\, ds = \int_1^{-2} (x^2 + x + 1)\, \sqrt{2}\, dx$$

$$= \sqrt{2}\left[\frac{1}{3}x^3 + \frac{1}{2}x^2 + x\right]_1^{-2}$$

$$= -\frac{9}{2}\sqrt{2}$$

Q3. Let D be the region lying inside both the sphere $x^2 + y^2 + z^2 = 4$ and the cylinder $x^2 + y^2 = 3$. Use the **divergence theorem** to find the outward flux $\iint_S (\mathbf{F} \cdot \mathbf{n})\, dS$ of the vector field $\mathbf{F} = y \sin z\ \mathbf{i} + yz^2\, \mathbf{j} - x^3\, \mathbf{k}$, where S is the boundary of D.

Solution:

In cylindrical coordinates the region D is given by

$$D = \left\{(r, \theta, z) | 0 \le r \le \sqrt{3}, 0 \le \theta \le 2\pi, -\sqrt{4 - r^2} \le z \le \sqrt{4 - r^2}\right\}.$$

The divergence of F is

$$\text{div } F = z^2.$$

By the divergence theorem,

$$\iint_S (\mathbf{F} \cdot \mathbf{n})\, dS = \iiint_D \text{div } F\, dV = \int_0^{2\pi} \int_0^{\sqrt{3}} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} z^2\, r\, dz\, dr\, d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} \int_0^{\sqrt{3}} [z^3]_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}}\, r\, dr\, d\theta$$

$$= \frac{2}{3} \int_0^{2\pi} \int_0^{\sqrt{3}} r(4 - r^2)^{\frac{3}{2}}\, dr\, d\theta$$

$$= \frac{2}{3} \int_0^{2\pi} d\theta \int_0^{\sqrt{3}} r(4 - r^2)^{\frac{3}{2}}\, dr$$

$$= -\frac{4\pi}{15} \left[(4 - r^2)^{\frac{5}{2}}\right]_0^{\sqrt{3}}$$

$$= \frac{124\pi}{15}$$

Q4.

a) Solve $z^2 = 2i$.

Solution:

With $r = 2$ and $\theta = \arg(2i) = \pi/2$, the polar form of the given number is

$$2i = 2(\cos\pi/2 + i\sin\pi/2).$$

We obtain the solutions of the equation

$$z_0 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 1 + i,$$

$$z_1 = \sqrt{2}\left(\cos\left(\frac{\frac{\pi}{2}+2\pi}{2}\right) + i\sin\left(\frac{\frac{\pi}{2}+2\pi}{2}\right)\right)$$

$$= \sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = -1 - i.$$

b) Find the **principal value** of $i^{(1-i)}$.

Solution: The principal value of $i^{(1-i)}$ is

$$e^{(i-1)\,\mathrm{Ln}\,i} = e^{(i-1)\left(\frac{i\pi}{2}\right)}$$

$$= e^{\frac{\pi}{2}+\frac{i\pi}{2}} = e^{\frac{\pi}{2}}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$$= ie^{\pi/2}.$$

Q5. Evaluate

$$\int_C \operatorname{Im}(z+2i)\,dz,$$

where C is the circular arc (in the first quadrant) along $|z| = 1$ from $z = 1$ to $z = i$.

Solution:

On curve C, we have $z = \cos t + i\sin t = e^{it}, 0 \le t \le \pi/2$, and $dz = ie^{it}dt$.

Thus,

$$\int_C \operatorname{Im}(z+2i)\,dz = \int_0^{\frac{\pi}{2}} (\operatorname{Im}(z)+2)\,ie^{it}dt$$

$$= \int_0^{\frac{\pi}{2}} (\sin t + 2)\,ie^{it}dt$$

$$= \int_0^{\frac{\pi}{2}} \left(\left(\frac{e^{it}-e^{-it}}{2i}\right)+2\right)\,ie^{it}dt$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{e^{2it}}{2} - \frac{1}{2} + 2ie^{it}\right)dt$$

$$= \left[\frac{e^{2it}}{4i} - \frac{t}{2} + 2e^{it}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{e^{i\pi}}{4i} - \frac{\pi}{4} + 2e^{\frac{i\pi}{2}} - \frac{1}{4i} - 2$$

$$= -\frac{1}{2i} - \frac{\pi}{4} + 2i - 2$$

$$= -2 - \frac{\pi}{4} + \frac{5}{2}i$$

Q6. Show that $f(z) = e^{2\bar{z}}$ is **nowhere** analytic.

Solution:

Let $z = x + i\,y$. We have

$$f(z) = e^{2\bar{z}} = e^{2x}(\cos 2y - i \sin 2y).$$

Put $u = e^{2x} \cos 2y$ and $v = e^{2x} \sin 2y$. We obtain

$$\frac{\partial u}{\partial x} = 2e^{2x} \cos 2y, \quad \frac{\partial v}{\partial y} = -2e^{2x} \sin 2y,$$

$$\frac{\partial u}{\partial y} = -2e^{2x} \cos 2y, \quad -\frac{\partial v}{\partial x} = 2e^{2x} \sin 2y,$$

Since the Cauchy-Riemann equations are not satisfied at any point, f is nowhere analytic.

Q7. Use **Cauchy's integral formulas** to evaluate

$$\oint_C \left[\frac{ze^{\pi z}}{(z-i)^2} + \frac{z+2}{z^2+2i\,z}\right] dz, \qquad \text{where C is the circle } |z-i| = 2.$$

Solution: Observe that the points $z = i$ and $z = 0$ are singularities lying within C. Let

$$I_1 = \oint_C \left[\frac{ze^{\pi z}}{(z-i)^2}\right] dz \quad and \quad I_2 = \oint_C \left[\frac{z+2}{z^2+2i\,z}\right] dz.$$

Let $f_1(z) = ze^{\pi z}$. We have $f_1{}'(z) = e^{\pi z}(1+\pi z)$, $f_1{}'(i) = e^{\pi i}(1+\pi i) = -1-\pi i$.
Using the Cauchy's integral formula, we obtain

$$I_1 = \oint_C \left[\frac{ze^{\pi z}}{(z-i)^2}\right] dz = 2\pi i f_1{}'(i) = 2\pi i(-1-\pi i) = 2\pi(\pi - i).$$

To evaluate I_2, put

$$f_2(z) = \frac{z+2}{z+2i}.$$

We have

$$f_2(0) = \frac{2}{2i} = -i.$$

So,

$$I_2 = \oint_C \left[\frac{z+2}{z^2+2i\,z}\right] dz = 2\pi i f_2(0) = 2\pi i\,(-i) = 2\pi.$$

Hence,

$$\oint_C \left[\frac{ze^{\pi z}}{(z-i)^2} + \frac{z+2}{z^2+2i\,z}\right] dz = I_1 + I_2 = 2\pi(\pi - i) + 2\pi = 2\pi(\pi + 1 - i).$$

Q8. Expand

$$f(z)=\frac{3z}{z^2-z-2}$$

in a **Laurent series** valid for the annular domain $0<|z+1|<3$.

Solution:

We want all powers of $(z+1)$ in the series. We obtain

$$f(z)=\frac{1}{z+1}+\frac{2}{z-2}=\frac{1}{z+1}+\frac{2}{-3+z+1}$$

$$=\frac{1}{z+1}-\frac{2}{3}\ \frac{1}{1-\frac{z+1}{3}}$$

$$=\frac{1}{z+1}-\frac{2}{3}\left(1+\frac{z+1}{3}+\frac{(z+1)^2}{3^2}+\frac{(z+1)^3}{3^3}+\cdots\right)$$

$$=\frac{1}{z+1}-\frac{2}{3}-\frac{2(z+1)}{3^2}-\frac{2(z+1)^2}{3^3}-\frac{2(z+1)^3}{3^4}-\cdots$$

The series valid for the annular domain $0<|z+1|<3$.

Q9. (a) Show that $z = 0$ is a **removable singularity** of

$$f(z) = \frac{2z - \sin 2z}{z^2}.$$

Solution: We know that Maclaurin series of $\sin z$ is given by

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

Replacing z by 2z gives

$$\sin 2z = 2z - \frac{(2z)^3}{3!} + \frac{(2z)^5}{5!} - \frac{(2z)^7}{7!} + \cdots$$

We obtain

$$\frac{2z - \sin 2z}{z^2} = \frac{2^3 z}{3!} - \frac{2^5 z^3}{5!} + \frac{2^7 z^5}{7!} - \cdots$$

From the form of the last series we see that $z = 0$ is a removable singularity.

(b) Find the **residue** at the essential singularity $z = 0$ of the function

$$f(z) = z^3 e^{1/z^2}.$$

Solution: Maclaurin series of e^z is given by

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

Replacing z by z^{-2} gives

$$e^{1/z^2} = 1 + z^{-2} + \frac{z^{-4}}{2!} + \frac{z^{-6}}{3!} + \cdots$$

Multiplying the result by z^3 obtains

$$z^3 e^{1/z^2} = z^3 + z + \frac{z^{-1}}{2!} + \frac{z^{-3}}{3!} + \cdots$$

The residue at $z = 0$ of the given function is $a_{-1} = 1/2$.

Q10. Use the **residue theorem** to evaluate

$$(a)\ \oint_C \csc z\ dz,$$ where C is the positively oriented circle $|z| = 1$.

Solution: Observe that

$$\csc z = \frac{1}{\sin z}$$

is not analytic at $z = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \cdots$. Since $z = 0$ the only singularity of the given function lying within or on C, we have

$$\int_C \csc z\ \ dz = 2\pi i \text{ Res}\,(\csc z\,, 0) = 2\pi i\ \lim_{z\to 0}\frac{z}{\sin z} = 2\pi i.$$

$$(b)\ \oint_C \frac{\cos z}{z^2(z-\pi)}\ dz,$$ where C is the positively oriented circle $|z-1| = 3$.

Solution: Observe that both singularities $z = 0$ are $z = \pi$ lie within C. Thus,

$$\oint_C \frac{\cos z}{z^2(z-\pi)}\ dz = 2\pi i\ [Res(\cos z\,, 0) + Res(\cos z\,, \pi)]$$

$$= 2\pi i\left[\lim_{z\to 0}\frac{d}{dz}\left(\frac{\cos z}{z-\pi}\right) + \lim_{z\to\pi}\frac{\cos z}{z^2}\right]$$

$$= 2\pi i\left[\lim_{z\to 0}\left(\frac{-(z-\pi)\sin z - \cos z}{(z-\pi)^2}\right) - \frac{1}{\pi^2}\right]$$

$$= 2\pi i\left(-\frac{1}{\pi^2} - \frac{1}{\pi^2}\right) = -\frac{4}{\pi}\ i$$