King Fahd University of Petroleum and Minerals

Department of Mathematics and Statistics

MATH 302 - Exam II (Term 132)

April 15, 2014

Duration: 120 Minutes

Name : Solution ID # : ______

Section # : ______ Serial #: ______

- Provide all **necessary steps** with **clear writing.**
- **Mobiles** and **calculators** are **NOT allowed** in the exam.

Question #	Marks	Maximum Marks
Q1		8
Q2		5
Q3		17
Q4		12
Q5		18
Q6		10
Q7		12
Q8		18
Total		**100**

Q1. Let C be the curve traced by the vector function $\mathbf{r}(t) = \langle \cos t, \sin t, 3t \rangle$, $0 \le t \le \pi$. Evaluate the length of the curve C.

Solution :

$$r'(t) = \langle -\sin t, \cos t, 3 \rangle$$

$$\|r'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 9}$$

$$= \sqrt{10}$$

The length of the curve C is

$$L(C) = \int_0^{\pi} \|r'(t)\| \, dt$$

$$= \int_0^{\pi} \sqrt{10} \, dt$$

$$= \sqrt{10}\,\pi$$

Q2. Is there a vector field $\mathbf{G}$ whose curl is given by curl $\mathbf{G} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$? *Justify your answer!*

Solution:

No, there is no such a vector field

since $\operatorname{div}(\operatorname{curl} G) = 1 + 1 + 1 = 3 \neq 0.$

Q3. Consider the surface S: $z = 2\sin^2(2x - y)$ and the point P$(\pi/2, \pi/4, 1)$.

(a) Find equations of the ***tangent plane*** and ***normal line*** to the surface S at P.

Solution: consider the function

$$F(x,y,z) = z - 2\sin^2(2x-y)$$

$$\nabla F(x,y,z) = \langle -8\sin(2x-y)\cos(2x-y),\ 4\sin(2x-y)\cos(2x-y),\ 1\rangle$$

$$\nabla F(\pi/2, \pi/4, 1) = \langle -8\sin\tfrac{3\pi}{4}\cos\tfrac{3\pi}{4},\ 4\sin\tfrac{3\pi}{4}\cos\tfrac{3\pi}{4},\ 1\rangle$$

$$= \langle 4, -2, 1\rangle$$

An equation for the tangent plane to S at P is

$$4\left(x - \tfrac{\pi}{2}\right) - 2\left(y - \tfrac{\pi}{4}\right) + (z-1) = 0$$

$$\Rightarrow \quad 4x - 2y + z = \tfrac{3\pi}{2} + 1$$

Parametric equations for the normal line are

$$x = \pi/2 + 4t$$

$$y = \pi/4 - 2t$$

$$z = 1 + t$$

(b) Find a ***unit vector*** that gives the direction in which $f(x,y) = 2\sin^2(2x - y)$ ***decreases*** most rapidly at Q$(\pi/2, \pi/4)$. What is the ***maximum rate*** of change of f at Q?

Solution:

$$\nabla f = \langle 8\sin(2x-y)\cos(2x-y),\ -4\sin(2x-y)\cos(2x-y)\rangle$$

$$\nabla f(\pi/2, \pi/4) = \langle -4, 2\rangle$$

$$|\nabla f(\pi/2, \pi/4)| = \sqrt{20} = 2\sqrt{5}.$$

f decreases most rapidly at Q in the direction of the unit vector

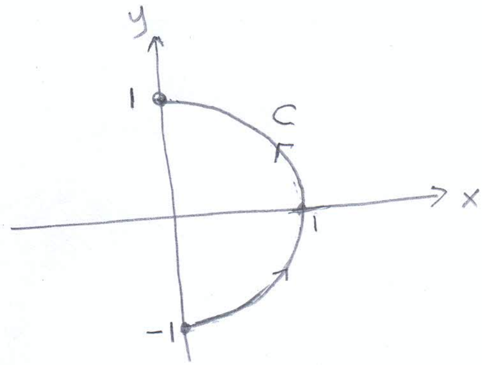
$$-\frac{\nabla f(\pi/2, \pi/4)}{|\nabla f(\pi/2, \pi/4)|} = \left\langle \tfrac{2}{\sqrt{5}}, -\tfrac{1}{\sqrt{5}}\right\rangle$$

The maximum rate of change of f at Q is $|\nabla f(\pi/2, \pi/4)|$

$$= 2\sqrt{5}.$$

Q4. Find the work done by the force $\mathbf{F}(x, y) = (2xy - x)\,\mathbf{i} + (x^2 + y^2 - y)\,\mathbf{j}$ acting along the curve $x^2 + y^2 = 1,\ x \geq 0,$ from $(0, -1)$ to $(0, 1)$.

Solution:



$$W = \int_C F \cdot dr$$

$C: \quad x = \cos t, \quad y = \sin t, \quad -\pi/2 \leq t \leq \pi/2$

$$W = \int_C (2xy - x)\,dx + (x^2 + y^2 - y)\,dy$$

$$= \int_{-\pi/2}^{\pi/2} (2\cos t \sin t - \cos t)(-\sin t\,dt) + (1 - \sin t)\cos t\,dt$$

$$= \int_{-\pi/2}^{\pi/2} (-2\cos t \sin^2 t + \cos t)\,dt$$

$$= \left[-\frac{2}{3}\sin^3 t + \sin t \right]_{-\pi/2}^{\pi/2}$$

$$= -\frac{2}{3} + 1 - \frac{2}{3} + 1 \quad = 2/3$$

Q5. *Show* that the line integral

$$\int_C 2xyz\,dx + (\cos y + x^2 z)\,dy + (1 + x^2 y)\,dz$$

is independent of path C between $(1,0,-1)$ and $(2,\pi/2,3)$. *Evaluate* the integral.

Solution:

Let $F = 2xyz\,\mathbf{i} + (\cos y + x^2 z)\,\mathbf{j} + (1+x^2 y)\,\mathbf{k}$

$$\text{Curl } F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz & \cos y + x^2 z & 1+x^2 y \end{vmatrix}$$

$$= \mathbf{i}(x^2 - x^2) - \mathbf{j}(2xy - 2xy) + \mathbf{k}(2xz - 2xz)$$

$$= \vec{0}$$

Thus, the line integral is independent of path.

Let ϕ be a potential function of F.

$$\phi_x = 2xyz \Rightarrow \phi = \int 2xyz\,dz$$

$$= x^2 yz + G(y,z)$$

$$\phi_y = \cos y + x^2 z = x^2 z + G_y(y,z)$$

$$\Rightarrow G_y(y,z) = \cos y$$

$$G(y,z) = \sin y + H(z)$$

$$\phi = x^2 yz + \sin y + H(z)$$

$$\phi_z = x^2 y + H'(z) = 1 + x^2 y \Rightarrow H'(z) = 1$$

$$H(z) = z$$

So, $\phi(x,y,z) = x^2 yz + \sin y + z.$

$$\int_C 2xyz\,dx + (\cos y + x^2 z)\,dy + (1+x^2 y)\,dz$$

$$= \phi(2, \pi/2, 3) - \phi(1,0,-1)$$

$$= (4)\left(\frac{\pi}{2}\right)(3) + \sin\frac{\pi}{2} + 3 - (-1) = 5 + 6\pi$$

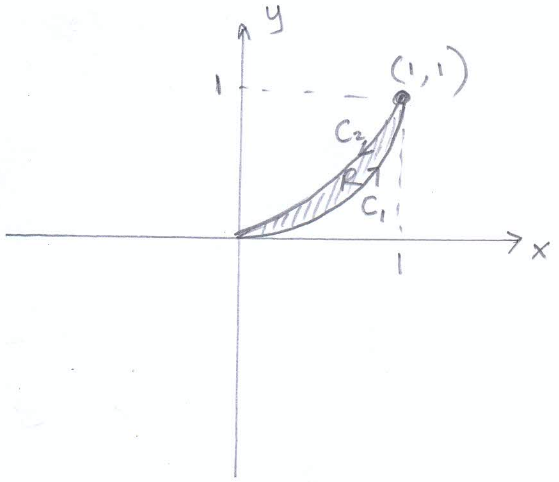
Q6. Let C be the positively oriented simple closed path given by $C = C_1 \cup C_2$, where

- C_1 is the portion of the graph of $y = x^3$ joining the points $(0,0)$ and $(1,1)$.
- C_2 is the portion of the graph of $y = x^2$ joining the points $(1,1)$ and $(0,0)$.

Use Green's theorem to evaluate

$$\oint_C (xy + \sin(x^3))\,dx + (x^2 - e^{-y^2})\,dy.$$

Solution:



Let $P = xy + \sin x^3$
$Q = x^2 - e^{-y^2}$.

$\frac{\partial P}{\partial y} = x$

$\frac{\partial Q}{\partial x} = 2x$.

Since $P, Q, \frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ are all continuous in R,

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_R (2x - x)\, dA$$

$$= \iint_R x\, dA$$

$$= \int_0^1 \int_{x^3}^{x^2} x\, dy\, dx$$

$$= \int_0^1 (x^3 - x^4)\, dx$$

$$= \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$$

Q7. Evaluate the surface integral

$$\iint_S \sqrt{1+4x^2+4y^2}\, dS\,;$$

S is the portion of the paraboloid $z = 4 - x^2 - y^2$ in the first octant ***outside*** the cylinder $x^2 + y^2 = 1$.

Solution:

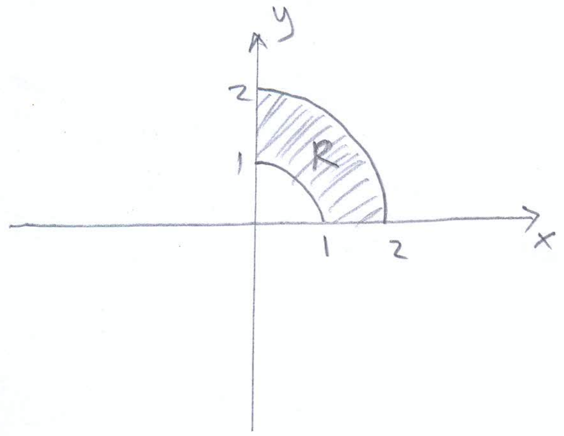
$$S: \quad z = f(x,y) = 4 - x^2 - y^2$$

$$dS = \sqrt{1+4x^2+4y^2}$$

$$\iint_S \sqrt{1+4x^2+4y^2}\, dS$$

$$= \iint_R (1+4x^2+4y^2)\, dA$$

$$= \int_0^{\pi/2}\int_1^2 (1+4r^2)\, r\, dr\, d\theta$$



$$= \left(\frac{\pi}{2}\right)\left(\frac{1}{2}r^2 + r^4\right)\Big|_1^2$$

$$= \left(\frac{\pi}{2}\right)\left(2+16-\frac{1}{2}-1\right)$$

$$= \frac{33\pi}{4}$$

Q8. Use Stokes' theorem to evaluate

$$\oint_C e^{x^2}\,dx + xy\,dy - yz\,dz,$$

where C is the trace of the cylinder $x^2 + y^2 = 4$ in the plane $x - 2y + z = -3$ oriented counterclockwise as viewed from above.

Solution:

Let $F = e^{x^2}\,i + xy\,j - yz\,k.$

$$\text{Curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{x^2} & xy & -yz \end{vmatrix} = -z\,i + y\,k$$

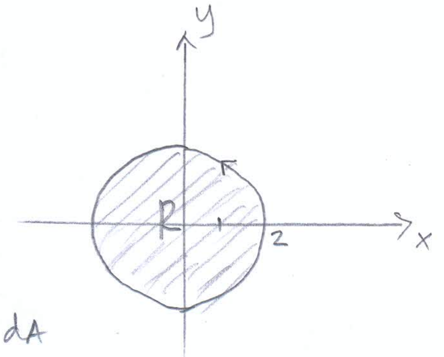
From the equation of the plane we get;

$$z = 2y - x - 3$$

$$dS = \sqrt{1+1+4} = \sqrt{6}$$

$$n = \frac{\langle 1, -2, 1 \rangle}{\sqrt{6}}.$$

By Stokes' Theorem,



$$\oint_C e^{x^2}\,dx + xy\,dy - yz\,dz$$

$$= \iint_S (\text{curl } F)\cdot n\,dS = \iint_R (-z+y)\,dA$$

$$= \iint_R (-2y + x + 3 + y)\,dA = \iint_R (x - y + 3)\,dA$$

$$= \int_0^{2\pi}\int_0^2 (r\cos\theta - r\sin\theta + 3)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{1}{3}r^3(\cos\theta - \sin\theta) + \frac{3}{2}r^2\right]_{r=0}^{r=2} d\theta = \int_0^{2\pi}\left[\frac{8}{3}(\cos\theta - \sin\theta) + 6\right] d\theta$$

$$= \left[\frac{8}{3}(\sin\theta + \cos\theta) + 6\theta\right]_0^{2\pi} = 12\pi$$