King Fahd University of Petroleum and Minerals

Department of Mathematics and Statistics

MATH 302 – Exam I (Term 132)

March 01, 2014

Duration: **120** Minutes

Name : SOLUTION KEY **ID #** :______________________

Section # : ___________ **Serial #:** __________

- Provide all **necessary steps** with **clear writing**.
- **Mobiles** and **calculators** are **NOT allowed** in the exam.

Question #	Marks	Maximum Marks
(1)		23
(2)		11
(3)		16
(4)		10
(5)		17
(6)		23
Total		**100**

(1) Let $\mathbf{S} = \{\langle a,b,c,d\rangle \mid a+b+c=d \text{ and } a,b,c,d \in \mathbb{R}\}$.

(a) Show that **S** is a ***subspace*** of $\mathbb{R}^4$.

Solution: **S** is nonempty since the vector $\mathbf{0} = \langle 0,0,0,0\rangle$ is in the set.

(i) Let $\mathbf{v_1} = \langle a_1,\ b_1,\ c_1,\ d_1\rangle$, $\mathbf{v_2} = \langle a_2,\ b_2,\ c_2,\ d_2\rangle$ be in **S.** Then

$$a_1 + b_1 + c_1 = d_1 \qquad \text{and} \qquad a_2 + b_2 + c_2 = d_2.$$

We have

$$\mathbf{v_1} + \mathbf{v_2} = \langle a_1 + a_2,\ b_1 + b_2,\ c_1 + c_2,\ d_1 + d_2\rangle \text{ and}$$

$$(a_1 + a_2) + (b_1 + b_2) + (c_1 + c_2) = (a_1 + b_1 + c_1) + (a_2 + b_2 + c_2) = d_1 + d_2.$$

So, $\mathbf{v_1} + \mathbf{v_2}$ is also in **S.**

(ii) Let k be a scalar. Then $k\mathbf{v_1} = \langle ka_1, kb_1, kc_1, kd_1\rangle$.

We have $ka_1 + kb_1 + kc_1 = k(a_1 + b_1 + c_1) = kd_1$. So, $k\mathbf{v_1}$ is also in **S.**

From (i) and (ii) we conclude that the set **S** is a subspace of $\mathbb{R}^4$.

(b) Find a ***basis*** and the ***dimension*** of **S**.

Solution: Let $\mathbf{v} = \langle a,b,c,d\rangle$ be in **S**. Since $a+b+c=d$, we have

$$\mathbf{v} = \langle a,b,c,d\rangle = \langle a,b,c,a+b+c\rangle = \langle a,0,0,a\rangle + \langle 0,b,0,b\rangle + \langle 0,0,c,c\rangle$$

$$= a\,\langle 1,0,0,1\rangle + b\,\langle 0,1,0,1\rangle + c\,\langle 0,0,1,1\rangle.$$

So, the set $\mathbf{B} = \{\langle 1,0,0,1\rangle, \langle 0,1,0,1\rangle, \langle 0,0,1,1\rangle\}$ spans **S**. Since matrix

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

has rank 3, the set **B** is linearly independent. Thus, **B** is a basis for **S**.
The dimension of **S** is 3 = the number of vectors in the basis **B**.

(c) Express the vector $\mathbf{v} = \langle 4,1,-2,3\rangle$ as a linear combination of vectors in the basis found in (b).

Solution: $\mathbf{v} = \langle 4,1,-2,3\rangle = 4\langle 1,0,0,1\rangle + \langle 0,1,0,1\rangle - 2\langle 0,0,1,1\rangle$.

(2) Solve the following system using ***Gaussian Elimination*** method:

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_1 + x_2 - x_3 &= 1 \\ x_1 - x_2 - x_3 &= 1. \end{aligned}$$

Solution:

Applying Gaussian Elimination to the augmented matrix of the system gives

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 \end{array}\right) \textit{Row operation} \Longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right).$$

We have

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_1 + x_2 &= 0 \\ x_3 &= 0. \end{aligned}$$

Thus, the solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

(3) Determine whether the given set of vectors is ***linearly independent*** or ***linearly dependent.***

(a) $\mathbf{u_1} = \langle 1, -1, 3, -1, 0 \rangle$, $\mathbf{u_2} = \langle 2, -1, -3, 1, 4 \rangle$, $\mathbf{u_3} = \langle -1, 0, 6, -2, -4 \rangle$.

Solution: The set of vectors is ***linearly dependent*** since $\mathbf{u_1} - \mathbf{u_2} - \mathbf{u_3} = \mathbf{0}$.

(b) $\mathbf{u_1} = \langle 1, -1, 3, -1 \rangle$, $\mathbf{u_2} = \langle 3, -3, 5, -4 \rangle$, $\mathbf{u_3} = \langle -2, 2, 0, 7 \rangle$.

Solution: Consider the matrix

$$A = \begin{pmatrix} 1 & -1 & 3 & -1 \\ 3 & -3 & 5 & -4 \\ -2 & 2 & 0 & 7 \end{pmatrix} \textit{Row operations} \Longrightarrow \begin{pmatrix} 1 & -1 & 3 & -1 \\ 0 & 0 & 1 & -1/4 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since $\operatorname{rank}(\boldsymbol{A}) = 3$ and the number of vectors is also 3, the given set is ***linearly independent***.

(4) Let $\mathbf{A}$ be a nonzero 5×7 matrix.

(a) What is the ***maximum rank*** that $\mathbf{A}$ can have?

Answer:

The ***maximum*** possible rank of $\mathbf{A}$ is the number of rows in $\mathbf{A}$, which is **5**.

(b) If $\operatorname{rank}(\mathbf{A}|\mathbf{B}) = 4$, then for what value(s) of $\operatorname{rank}(\mathbf{A})$ is the system $\mathbf{AX} = \mathbf{B}$, $\mathbf{B} \neq \mathbf{0}$, inconsistent? Consistent?

Answer:

The system is ***inconsistent*** if $\operatorname{rank}(\mathbf{A}) < \mathbf{4}$.

The system is ***consistent*** if $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{A}|\mathbf{B}) = \mathbf{4}$.

(c) If $\operatorname{rank}(\mathbf{A}) = 3$, then how many ***parameters*** does the solution of the system $\mathbf{AX} = \mathbf{0}$ have?

Answer:

The system has $n = 7$ unknowns and the rank of A is $r = 3$. Thus, the solution of the system has $n - r = \mathbf{4}$ parameters.

(5) (**a**) Find matrix **A** if $\mathbf{A}^{-1} = \begin{pmatrix} 2 & -1 & 2 \\ 1 & -1 & 2 \\ -3 & 2 & -3 \end{pmatrix}$.

Solution: To find **A** we compute $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$:

$$\left(\begin{array}{ccc|ccc} 2 & -1 & 2 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 & 0 \\ -3 & 2 & -3 & 0 & 0 & 1 \end{array}\right) \quad R_{12} \Longrightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 1 & 0 \\ 2 & -1 & 2 & 1 & 0 & 0 \\ -3 & 2 & -3 & 0 & 0 & 1 \end{array}\right)$$

$$-2R_1 + R_2 \Rightarrow 3R_1 + R_3 \Rightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 & -2 & 0 \\ 0 & -1 & 3 & 0 & 3 & 1 \end{array}\right)$$

$$R_2 + R_3 \Rightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right)$$

$$2R_3 + R_2 \Rightarrow -2R_3 + R_1 \Rightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & -2 & -1 & -2 \\ 0 & 1 & 0 & 3 & 0 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right)$$

$$R_1 + R_2 \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 3 & 0 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right)$$

We have $\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ 3 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$.

(**b**) Solve the system $\mathbf{AX} = \mathbf{B}$, where **A** is the matrix found in (a),

$\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$.

Solution: The solution is $\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{A}^{-1}\mathbf{B} = \begin{pmatrix} 2 & -1 & 2 \\ 1 & -1 & 2 \\ -3 & 2 & -3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -3 \\ -5 \\ 5 \end{pmatrix}$.

(6) Let $\mathbf{A} = \begin{pmatrix} -1 & 2 & -2 \\ 2 & -1 & 2 \\ -2 & 2 & -1 \end{pmatrix}$.

(a) Verify that eigenvalues of **A** are $\lambda_1 = -5$ and $\lambda_2 = \lambda_3 = 1$.

Solution: Characteristic equation of **A** is

$$\begin{vmatrix} -1-\lambda & 2 & -2 \\ 2 & -1-\lambda & 2 \\ -2 & 2 & -1-\lambda \end{vmatrix} = (-1-\lambda)(\lambda^2+2\lambda-3) - 4(-2-2\lambda+4)$$

$$= \lambda^3 + 3\lambda^2 - 4\lambda + 5 = (\lambda+5)(\lambda-1)^2 = 0.$$

Thus, eigenvalues of **A** are $\lambda_1 = -5$ and $\lambda_2 = \lambda_3 = 1$.

(b) Find an ***orthogonal matrix*** **P** that diagonalizes **A** and find the diagonal matrix $\mathbf{D} = \mathbf{P}^T\mathbf{A}\mathbf{P}$.

Solution: For $\lambda_1 = -5$, we have

$$\left(\begin{array}{ccc|c} 4 & 2 & -2 & 0 \\ 2 & 4 & 2 & 0 \\ -2 & 2 & 4 & 0 \end{array}\right) \textit{ Row operation } \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The corresponding eigenvector is $K_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

For $\lambda_2 = \lambda_3 = 1$, we have

$$\left(\begin{array}{ccc|c} -2 & 2 & -2 & 0 \\ 2 & -2 & 2 & 0 \\ -2 & 2 & -2 & 0 \end{array}\right) \textit{ Row operation } \Rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Two corresponding orthogonal eigenvectors are

$$K_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

The required orthogonal matrix is $P = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \\ -1/\sqrt{3} & 2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \end{pmatrix}$

The diagonal matrix is $D = \begin{pmatrix} -5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.