KFUPM, DEPARTMENT OF MATHEMATICS AND STATISTICS

MATH 280: FINAL EXAM, SEMESTER (132), MAY 17, 2014

Time: 08:00 to 11:00 am

Name : ....

ID : .... Section : ....

Exercise	**Points**
1	____ / 12
2	____ / 10
3	____ / 10
4	____ / 10
5	____ / 12
6	____ / 12
7	____ / 12
8	____ / 12
9	____ / 12
10	____ / 14
11	____ / 12
12	____ / 12
Total	____ / 140

Exercise 1 (12 pts)**.** Let

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}.$$

(1) For each real number t, find the matrix $\exp(tA)$ (exponential of tA).

(2) Solve the following system of differential equations:

$$\begin{cases} x'(t) = & 3x(t) & +y(t) & +z(t) \\ y'(t) = & & 3y(t) & +z(t) \\ z'(t) = & & & 3z(t) \end{cases}$$

with $x(0) = y(0) = z(0) = 1$.

Exercise 2 (10 pts)**.** Let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

find a real matrix B such that $B^2 = A$.

Exercise 3 (10 pts).

(1) Find the distance between the point $P_0 = (1, 2)$ and the line L of equation $y = 3x$.

(2) Find the closest point Q of L to the point $P_0 = (1, 2)$.

Exercise 4 (10 pts)**.** Let $S = \{(x + 2y, 2x - y, 2x + 2y)^T : x, y \in \mathbb{R}\}$.

(1) Find an orthonormal basis of the vector space S (use Gram-Schmidt).

(2) Find the closest vector $\mathbf{p}$ of S to $\mathbf{w} = (1, 0, 1)^T$.

Exercise 5 (12 pts)**.** Find a QR-decomposition of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Exercise 6 (12 pts)**.**

(1) Find the transition matrix from $\mathcal{B} = (1, x, x^2)$ to the basis basis $\mathcal{B}' = (1, 1 - x, 1 + x + x^2)$.

(2) Given any $p(x) = a + bx + cx^2$ in $\mathbf{P}_2$, find the coordinates of $p(x)$ with respect to the basis $\mathcal{B}' = (1, 1 - x, 1 + x + x^2)$.

Exercise 7 (12 pts)**.** Let V be a real vector space with dimension 3 and $\mathcal{B} = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ be a basis of V.

Let $T : V \longrightarrow V$ be the linear operator with matrix relatively to $\mathcal{B}$:

$$M = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 2 & 2 \\ 2 & 3 & 1 \end{pmatrix}$$

(1) Set $\mathbf{v}_1 = \mathbf{u}_1 + \mathbf{u}_2$, $\mathbf{v}_2 = \mathbf{u}_1 - \mathbf{u}_2$ and $\mathbf{v}_3 = \mathbf{u}_1 + \mathbf{u}_3$. Show that $\mathcal{B}' = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is a basis of V.

(2) Find the matrix of T with respect to $\mathcal{B}'$.

Exercise 8 (12 pts)**.** Let A be a 2×2-matrix such that $\mathrm{tr}(A) = 12$ and $\det(A) = 27$. Find the eigenvalues of A. Is A diagonalizable?

Exercise 9 (12 pts)**.** Let A be an $n \times n$-matrix and $B = 6I_n - 5A + A^2$ and λ be an eigenvalue of A.

(1) Show that if $\mathbf{x}$ is an eigenvector of A associated to λ, then $\mathbf{x}$ is an eigenvector of B associated to an eigenvalue μ. How are λ and μ related?

(2) Show that if $\lambda = 2$, then B is a singular matrix.

Exercise 10 (14 pts)**.** Find an orthogonal matrix that diagonalizes the symmetric matrix

$$A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}.$$

(the eigenvalues of A are $\lambda_1 = \lambda_2 = 2$ and $\lambda_3 = 8$)

Exercise 11 (12 pts)**.** Find the critical points of the function

$$f(x, y) = 4xy - x^4 - y^4$$

and classify them as relative maxima, relative minima, or saddle points.

Exercise 12 (12 pts)**.** Express the quadratic equation

$$9x^2 - 4xy + 6y^2 - 10x - 20y = 5$$

in the matrix form

$$\mathbf{x}^T A\mathbf{x} + \mathbf{K}\mathbf{x} + f = 0,$$

where $\mathbf{x}^T A\mathbf{x}$ is the associated quadratic form and $\mathbf{K}$ is an appropriate matrix.

Identify the conic section represented by the given equation.