King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
(Math 202)

First Exam
Term 132
Saturday, March 01, 2014
Net Time Allowed: 120 minutes

Name:	
ID:	
Section No:	

Key

Instructions:

1. Calculators and Mobiles are not allowed.
2. Provide detailed answers. You may lose points for messy work.
3. No points for answers without justification.
4. Make sure that you have 8 questions (one in each page).

Q.	Points	Maximum Points
1		8
2		12
3		14
4		12
5		14
6		14
7		14
8		12
Total		**100**

(1) Verify that $y = \frac{1}{4 - x^2}$ is a solution to the differential equation: [8 points]

$$y' = 2xy^2$$

Sol:

$$y' = -(4-x^2)^{-2}(-2x)$$

$$= \frac{2x}{(4-x^2)^2} \quad 4$$

Now consider the RHS of D.E.:

$$2xy^2 = 2x\left(\frac{1}{4-x^2}\right)^2 \quad 2$$

$$= y'$$

So, $y = \frac{1}{4-x^2}$ is a Sol. 2

(2) Let $y = e^{-x} + c_1 x + c_2$, be a two parameter family of solutions of $\dfrac{d^2y}{dx^2} = e^{-x}$. Find the solution of the differential equation that satisfies the initial conditions $y(1) = 1$ and $y'(1) = 4$.

[12 points]

Sol:

Step-I: $y = e^{-x} + c_1 x + c_2$

$y' = -e^{-t} + c_1$ 2

Step-II: Use initial conditions:

$y(1) = 1 \implies 1 = e^{-1} + c_1 + c_2$ —— 1

$y'(1) = 4 \implies 4 = -e^{-1} + c_2$ —— 2 3

Step-III: Solve 1 & 2 together

$4 + 1/e = c_1$

$-(3 + \frac{2}{e}) = c_2$ 5

So $y = e^{-x} + (4 + e^{-1})x - (3 - 2e^{-1})$ 2

(3) [14 points]

a) Show that the following differential equation is separable.

$$\frac{dy}{dx} = \frac{xy+3x-y-3}{xy-2x+4y-8}$$

6/14

Sol: $\frac{dy}{dx} = \frac{x(y+3)-(y+3)}{x(y-2)+4(y-2)}$

$= \frac{(y+3)(x-1)}{(y-2)(x+4)}$ 4

So $\frac{dy}{dx} = f(x)\,g(y)$ where

$f(x) = \frac{x-1}{x+4}$ 1 & $g(y) = \frac{y+3}{y-2}$ 1

b) Solve the following Initial Value Problem by separation of variables

$$\sqrt{1-y^2}\,dx - \sqrt{1-x^2}\,dy = 0, \qquad y(0) = \frac{\sqrt{3}}{2}$$

8/14

$\sqrt{1-y^2}\,dx = \sqrt{1-x^2}\,dy \Leftrightarrow \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$ 2

$\sin^{-1} y = \sin^{-1}(x) + C$ 2 $\Leftrightarrow y = \sin(\sin^{-1}(x) + c)$

$y(0) = \sin(\sin^{-1}(0) + c) = \sqrt{3}/2$ 2

$\Rightarrow \sin(c) = \frac{\sqrt{3}}{2} \Rightarrow c = \pi/3$

$y = \sin(\sin^{-1}(x) + \pi/3)$ 2

(4) Solve the following Initial Value Problem: [12 points]

$$\frac{dx}{dy}+(\cot y)x=\sin y,\quad x(\pi/2)=0$$

The D.E. is Linear in x, so the Integrating factor is 1

$$u(y)=e^{\int \cot y\, dy}=e^{\ln|\sin y|} \quad 3$$

$$=|\sin y|$$

Now take $u(y)=\sin y$, multiply

$$\sin y\frac{dx}{dy}+(\cos y)x=\sin^2 y \quad 2$$

$$\Leftrightarrow (x\sin y)'=\sin^2 y \quad 2$$

$$x\sin y=\frac{1}{2}\int(1-\cos 2y)\,dy$$

$$=\frac{1}{2}y-\frac{1}{4}\sin(2y)+C \quad 2$$

$$\int \cot x = \int \frac{\cos x}{\sin x}\,dx \quad 1$$

I.C.: $x\left(\frac{\pi}{2}\right)=0$ gives $0=\frac{\pi}{4}-0+C \Rightarrow C=\frac{-\pi}{4}$

$$\boxed{x\sin y=\frac{1}{2}y-\frac{1}{4}\sin(2y)-\frac{\pi}{4}} \quad 1$$

(5) Solve the Initial Value Problem, [14 points]

$$\frac{dy}{dx} = \frac{3 - ye^{-2y}\sin(xy) + 4xy}{e^{-2y}\left[2\cos(xy) + x\sin(xy)\right] - 2x^2},$$

Subject to $y(0) = 0$

Sol: Rewrite:

$$\underbrace{\left(3 - ye^{-2y}\sin(xy) + 4xy\right)}_{m}dx + \left(2x^2 - \underbrace{e^{-2y}\left(2\cos(xy) + x\sin(xy)\right)}_{n}\right)dy = 0 \qquad 2$$

$$\frac{\partial m}{\partial y} = -e^{-2y}\sin(xy) + 2ye^{-2y}\sin(xy) - xye^{-2y}\cos(xy)$$

$$= \frac{\partial n}{\partial x} \Rightarrow \text{Exact} \qquad 3$$

$$f(x,y) = c_0, \quad \frac{\partial f}{\partial x} = m = 3 - ye^{-2y}\sin(xy) + 4xy$$

$$\Rightarrow f(x,y) = 3x + \frac{y}{y}e^{-2y}\cos(xy) + 2x^2y + h(y) \qquad 3$$

$$\frac{\partial f}{\partial y} = -2e^{-2y}\cos(xy) - xe^{-2y}\sin(xy) + 2x^2 + \frac{\partial h}{\partial y} \qquad 2$$

$$\frac{\partial h}{\partial y} = 0 \Rightarrow h(y) = c_2$$

So $2x^2y + 3x + e^{-2y}\cos(xy) + c = 0$ 2

$y(0) = 0 \Rightarrow \quad 0 + 0 + 1 + c = 0 \Rightarrow c = -1$ 2

$$\boxed{2x^2y + 3x + e^{-2y}\cos(xy) - 1 = 0}$$

(6) Given the nonlinear first order, differential equation $xy\,dx + \left(2x^2 + 3y^2 - 20\right)dy = 0$

a) Show that the differential equation is not exact. [14 points]

2 2

$M = xy$ and $N = 2x^2 + 3y^2 - 20$

$M_y = x \neq N_x = 4x$

6/14

2

b) Find an appropriate integrating factor that makes it exact.

8/14

2 $$\frac{M_y - N_x}{N} = \frac{x - 4x}{2x^2 + 3y^2 - 20} = \frac{-3x}{2x^2 + 3y^2 - 20}$$

(It depends on Both x and y)

$$\frac{N_x - M_y}{M} = \frac{4x - x}{xy} = 3/y \text{ , fun. of } y \text{ only!}$$

So $\mu = e^{\int \frac{3}{y} dy} = e^{3\ln y} = y^3$, Now 3

$xy^4\,dx + (2x^2y^3 + 3y^5 - 20y^3) = 0$ is 3

an Exact One!

(7) Use an appropriate substitution to transform the given differential equation to a linear differential equation. [14 points]

a) $2\frac{dy}{dx}+(\tan x)y-\frac{(4x+5)}{\cos x}y^3=0$

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First, re-write it in S.F.

$$\frac{dy}{dx}+\frac{1}{2}\tan(x)\,y=\frac{(4x+5)}{2\cos x}y^3$$ 2

Bernoulli, $n=3$

$P(x)=\frac{1}{2}\tan x$ & $Q(x)=\frac{4x+5}{2\cos x}$ 2

let $u=y^{-2} \Rightarrow u^{-1/2}=y \Rightarrow \frac{dy}{dx}=-\frac{1}{2}u^{-\frac{3}{2}}\frac{du}{dx}$ 2

substit. in D.E : $-\frac{1}{2}u^{-3/2}\frac{du}{dx}+\frac{1}{2}u^{-1/2}\tan x=\frac{4x+5}{2\cos x}u^{-\frac{3}{2}}$

Multiply by $u^{3/2} \Rightarrow$ $\boxed{-\frac{1}{2}\frac{du}{dx}+\frac{1}{2}(\tan x)u=\frac{4x+5}{2\cos x}}$ 2

Linear in u

b) $\frac{dy}{dx}=\tan^2(x+y)$

6/14

let $u=x+y \Rightarrow \frac{du}{dx}=1+\frac{dy}{dx}$ 2

So $\frac{du}{dx}=1+\tan^2 u$ 2

So $\frac{dx}{du}+0.x=1+\tan^2 u$ 2

Linear in x.

(8) A culture initially has N_o bacteria at time $t=0$ and N_1 bacteria at time $t=1$, where

$N_1 = \frac{3}{2} N_0$ [12 points]

If the growth of the bacteria at time t is proportional to the number of bacteria at time t, **then determine the time necessary for Bacteria to triple.**

Sol: Let $N(t)$ = # of Bacteria at time

$$\frac{dN}{dt} = kN \quad 2$$

$$\text{So } N(t) = Ae^{kt} \quad 3$$

$$N_0 = Ae^{0t} = A \quad 2$$

$$\text{So } N(t) = N_0 e^{kt}$$

$$N(1) = \frac{3}{2} N_0 \quad 2$$

So $\frac{3}{2} N_0 = N_0 e^{k}$, therefore: $\frac{3}{2} N_0 = N_0 e^{k}$

$$N(t) = N_0 \left(\frac{3}{2}\right)^t \quad 1$$

Want to find T so that $N(T) = 3N_0$

$$N(T) = N_0 (3/2)^T = 3N_0$$

$\left(\frac{3}{2}\right)^T = 3$ so $T \ln\left(\frac{3}{2}\right) = \ln 3$

$$\boxed{T = \frac{\ln 3}{\ln(1.5)}} \quad 2$$