Name:

ID#:

Quiz 3

Q1. Write the form of the partial fraction decomposition of:

$$\frac{x}{(x-1)(x^2+x+1)^2\underbrace{(x^2-1)}_{(x-1)(x+1)}}$$

[Do not evaluate the coefficients]

so, the denominator is $(x-1)^2(x^2+x+1)^2(x+1)$

$$\frac{x}{(x-1)^2(x^2+x+1)^2(x+1)} = \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} + \frac{A_3x+A_4}{x^2+x+1} + \frac{A_5x+A_6}{(x^2+x+1)^2} + \frac{A_7}{x+1}$$

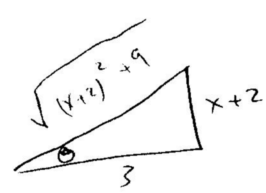
Q2. Find $J = \int \frac{x^2+5x+14}{x^2+4x+13}\,dx$

$$\begin{array}{r|l} & 1 \\ x^2+4x+13 & x^2+5x+14 \\ \mp & x^2+4x+13 \\ \hline & x+1 \end{array}$$

$$\Rightarrow \quad J = \int \left[1 + \frac{x+1}{x^2+4x+13}\right] dx$$

$$= x + \underbrace{\int \frac{x+1}{x^2+4x+13}\,dx}_{K}$$

$$K = \int \frac{x+1}{x^2+4x+13}\,dx = \int \frac{x+1}{(x+2)^2+9}\,dx$$



Let $x+2 = 3\tan\theta$, then $x = 3\tan\theta - 2$

$dx = 3\sec^2\theta\, d\theta$

$$K = \int \frac{3\tan\theta - 1}{\underbrace{9\tan^2\theta + 9}_{9\sec^2\theta}} \cdot 3\sec^2\theta\, d\theta = \frac{3}{9}\int (3\tan\theta - 1)\, d\theta$$

$$= \frac{1}{3}\left[-3\ln|\cos\theta| - \theta\right] + C$$

$$= -\ln\left|\frac{3}{\sqrt{(x+2)^2+9}}\right| - \tan^{-1}\left(\frac{x+2}{3}\right) + C.$$

Q3. Find if the following integrals are convergent or divergent:

a. $I = \int_5^\infty \frac{2+\cos x}{x}\,dx$

$-1 \le \cos x \le 1$,

$$\frac{1}{x} \le \frac{2+\cos x}{x} \Rightarrow \underbrace{\int_5^\infty \frac{dx}{x}}_{\text{divergent}} \le \int_5^\infty \frac{2+\cos x}{x}\,dx$$

By the direct comparison test, I is divergent.

b. $J = \int_5^\infty \frac{2+\cos x}{x^2}\,dx$

$$\frac{2+\cos x}{x^2} \le \frac{3}{x^2} \Rightarrow \int_5^\infty \frac{2+\cos x}{x^2}\,dx \le \underbrace{\int_5^\infty \frac{3\,dx}{x^2}}_{\text{convergent}}$$

By the direct comparison test, J is convergent.

Q4. Find $K = \int \frac{dx}{x\sqrt{9-(\ln x)^2}}$, $0 < x < e^3$.

$u = \ln x \Rightarrow K = \int \frac{du}{\sqrt{9-u^2}} = \sin^{-1}\left(\frac{u}{3}\right) + c$

$= \sin^{-1}\left(\frac{\ln x}{3}\right) + c.$

Q5. Find $M = \int_0^{\frac{3\pi}{2}} \frac{\sin x}{\sqrt{1+\cos x}}\, dx$

- discontinuous (vertical asym) at $x = \pi$.
- $M = \underbrace{\int_0^{\pi} \frac{\sin x}{\sqrt{1+\cos x}}\, dx}_{I} + \underbrace{\int_{\pi}^{\frac{3\pi}{2}} \frac{\sin x\, dx}{\sqrt{1+\cos x}}}_{J}$
- We have: $\int \frac{\sin x}{\sqrt{1+\cos x}}\, dx = -2\sqrt{1+\cos x} + c$ (clear, by substitution $u^2 = 1+\cos x$)
- $I = \lim_{t\to\pi^-} \int_0^t \frac{\sin x}{\sqrt{1+\cos x}}\, dx = \lim_{t\to\pi^-} \left[-2\sqrt{1+\cos x}\right]_0^t$

  $= \lim_{t\to\pi^{-1}} \left[-2\sqrt{1+\cos t} + 2\sqrt{2}\right] = \boxed{2\sqrt{2}}$

- $J = \lim_{s\to\pi^+} \int_s^{\frac{3\pi}{2}} \frac{\sin x}{\sqrt{1+\cos x}}\, dx = \lim_{s\to\pi^+} \left[-2\sqrt{1+\cos x}\right]_s^{\frac{3\pi}{2}}$

  $= \lim_{s\to\pi^+} \left[-2 + 2\sqrt{1+\cos s}\right] = \boxed{-2}$

$M = 2\sqrt{2} - 2$

Bonus:

Find if the following integral convergent or divergent:

$$J = \int_0^1 \frac{dt}{t - \sin t}$$

- this is an improper integral type 2; we cannot integrate the function $\frac{1}{t-\sin t}$, and we cannot use the comparison because it is type 2 and the comparison tests are for type 1.

- The idea is to "transform" the integral to type 1 using the substitution: $t = \frac{1}{y}$, i.e. $y = \frac{1}{t}$, $dt = \frac{-dy}{y^2}$

$t = 0^+ \to y = +\infty$

$t = 1 \to y = 1$

$$\Rightarrow J = \int_\infty^1 \frac{\frac{-dy}{y^2}}{\frac{1}{y} - \sin\frac{1}{y}} = \int_1^\infty \frac{dy}{y - y^2 \sin\frac{1}{y}}$$

- Notice that $y^2 \sin\frac{1}{y} \geq 0$ $\left(y^2 \geq 0, \text{ and } \sin\frac{1}{y} > 0 \text{ as } 0 < \frac{1}{y} < \frac{\pi}{2} \right)$

So: $y - y^2 \sin\frac{1}{y} \leq y$

So: $\frac{1}{y} \leq \frac{1}{y - y^2 \sin\frac{1}{y}}$, i.e. $\underbrace{\int_1^\infty \frac{dy}{y}}_{\text{divergent}} \leq \int_1^\infty \frac{dy}{y - y^2 \sin\frac{1}{y}}$

$\Downarrow$

J is divergent by the direct comparison test.