King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 102 - Exam II - Term 132

Duration: 120 minutes

KEY

Name:__________ ID Number:__________

Section Number:__________ Serial Number:__________

Class Time: __________ Instructor's Name:__________

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write neatly and eligibly. You may lose points for messy work.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 7 pages of problems (Total of 7 Problems)

Page Number	Points	Maximum Points
1		24
2		14
3		10
4		12
5		16
6		10
7		14
Total		100

1. a) (**6 points**) Evaluate $\int_1^9 \frac{3^{\sqrt{x}}}{\sqrt{x}}\,dx$.

Let $u=\sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}}\,dx$ & $x=1 \Rightarrow u=1$, $x=9 \Rightarrow u=3$ 1

$$\int_1^9 \frac{3^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int_1^3 3^u\,du = 2\cdot \frac{3^u}{\ln 3}\Big]_1^3 = \frac{2}{\ln 3}\left(3^3-3\right) = \frac{48}{\ln 3}$$

2 2 1

b) (**6 points**) Find the form of the partial fraction decomposition for $\frac{x^5}{x^4-1}$.
Do not find the numerical values of the coefficients.

$$\frac{x^5}{x^4-1} = x + \frac{x}{x^4-1} \quad 2$$

$$= x + \frac{x}{(x-1)(x+1)(x^2+1)} \quad 1$$

$$= x + \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1} \quad 1+1+1$$

c) (**6 points**) If $\sinh x = \frac{4}{3}$, then find the value of $\sinh(2x)$.

- $\cosh^2 x - \sinh^2 x = 1$ 1

$\Rightarrow \cosh^2 x = 1+\sinh^2 x = 1+\left(\frac{4}{3}\right)^2 = \frac{25}{9}$

$\Rightarrow \cosh x = \frac{5}{3}$ 2 (as $\cosh x > 0$ for all x)

- $\sinh(2x) = 2\sinh x \cosh x$ 2

$= 2\cdot\frac{4}{3}\cdot\frac{5}{3}$

$= \frac{40}{9}$ 1

d) (**6 points**) Find $\int \tan^3 x \cos^5 x\,dx$.

$$\int \tan^3 x \cos^5 x\,dx = \int \sin^3 x \cos^2 x\,dx \quad 2$$

$$= \int (1-\cos^2 x)\cos^2 x \cdot \sin x\,dx \quad 1 \qquad \begin{bmatrix} u=\cos x \\ du=-\sin x\,dx \end{bmatrix}$$

$$= -\int (1-u^2)u^2\,du \quad 1$$

$$= -\int u^2-u^4\,du$$

$$= -\left(\tfrac{1}{3}u^3-\tfrac{1}{5}u^5\right)+C \quad 1$$

$$= -\tfrac{1}{3}\cos^3 x + \tfrac{1}{5}\cos^5 x + C \quad 1$$

(3)

e) (**6 points**) Find $\int \frac{x^2}{e^{2x}}\,dx$.

x^2	$(+)\; e^{-2x}$
$2x$	$(-)\; -\frac{1}{2}e^{-2x}$
2	$(+)\; \frac{1}{4}e^{-2x}$
0	$-\frac{1}{8}e^{-2x}$

$$\cdot \int x^2 e^{-2x}$$

$$= -\tfrac{1}{2}x^2 e^{-2x} - \tfrac{1}{2}x e^{-2x} - \tfrac{1}{4}e^{-2x} + C \quad (3)$$

OR

$$u = x^2 \qquad dv = e^{-2x}dx \quad (1)$$

$$du = 2x\,dx \qquad v = -\tfrac{1}{2}e^{-2x}\,dx \quad (1)$$

$$\int x^2 e^{-2x}dx = -\tfrac{1}{2}x^2 e^{-2x} + \int x e^{-2x}dx \quad (1)$$

$$\boxed{\begin{array}{ll} u = x & dv = e^{-2x}dx \\ du = dx & v = -\tfrac{1}{2}e^{-2x} \end{array}} \quad (1)$$

$$= -\tfrac{1}{2}x^2 e^{-2x} - \tfrac{1}{2}x e^{-2x} + \tfrac{1}{2}\int e^{-2x}dx \quad (1)$$

$$= -\tfrac{1}{2}x^2 e^{-2x} - \tfrac{1}{2}x e^{-2x} - \tfrac{1}{4}e^{-2x} + C. \quad (1)$$

2. (**8 points**)Find the slope of the tangent line to the curve $y = \cosh^4 x - \sinh^4 x$ at $x = \ln 2$. Write your answer in the form $\frac{p}{q}$, where p and q are nonzero integers.

APP

$$\cdot\; y' = 4\cosh^3 x \cdot \sinh x - 4\sinh^3 x \cdot \cosh x \quad (1+1)$$

$$= 4\cosh x \cdot \sinh x\,(\cosh^2 x - \sinh^2 x)$$

$$= 4\cosh x \cdot \sinh x \cdot 1$$

$$= 4\cosh x \cdot \sinh x \quad (2)$$

$$\cdot\; \text{Slope} = y'\big|_{x=\ln 2} = 4\cosh(\ln 2)\cdot\sinh(\ln 2)$$

$$= 4 \cdot \frac{e^{\ln 2} + e^{-\ln 2}}{2} \cdot \frac{e^{\ln 2} - e^{-\ln 2}}{2} \quad (1+1)$$

$$= (2+\tfrac{1}{2})\cdot(2-\tfrac{1}{2}) \quad (1)$$

$$= \frac{5}{2}\cdot\frac{3}{2}$$

$$= \frac{15}{4} \quad (1)$$

3. (**10 points**) Evaluate $\int \frac{x^2 + x}{\sqrt{4 - x^2}}\, dx$.

2 Let $x = 2\sin\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

1 Then $\cdot\ dx = 2\cos\theta\, d\theta$

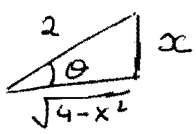
1 $\cdot\ \sqrt{4-x^2} = \sqrt{4 - 4\sin^2\theta} = 2\sqrt{\cos^2\theta} = 2|\cos\theta| = 2\cos\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$\int \frac{x^2+x}{\sqrt{4-x^2}}\, dx = \int \frac{4\sin^2\theta + 2\sin\theta}{2\cos\theta} \cdot 2\cos\theta\, d\theta$$

$$= \int \left(4\sin^2\theta + 2\sin\theta\right) d\theta \qquad 1$$

$$= \int \left(2\left(1 - \cos(2\theta)\right) + 2\sin\theta\right) d\theta \qquad 1$$

$$= 2\theta - \sin(2\theta) - 2\cos\theta + C \qquad 1$$



$\sin\theta = \frac{x}{2}$

$\cos\theta = \frac{\sqrt{4-x^2}}{2}$

$$= 2\sin^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}x\sqrt{4-x^2} - \sqrt{4-x^2} + C$$

1 1 1

4. (**12 points**) Evaluate $\int \frac{2x^3 - 4x^2 - 2}{(x^2+1)(x-1)^2}\,dx$.

$$\frac{2x^3-4x^2-2}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$ $\boxed{3 = 1+1+1}$

1

$$2x^3-4x^2-2 = (Ax+B)(x-1)^2 + C(x-1)(x^2+1) + D(x^2+1)$$

$x=1$: $-4 = 0+0+2D \Rightarrow \boxed{D=-2}$

$x=0$: $-2 = B - C + D \Rightarrow B - C = 0 \Rightarrow \boxed{B=C}$

$x=-1$: $-8 = 4(B-A) - 4C + 2D \xRightarrow{D=-2} -4 = 4B - 4A - 4C \xRightarrow{B=C} -4 = -4A \Rightarrow \boxed{A=1}$

$x=2$: $-2 = (2A+B) + 5C + 5D \xRightarrow[D=-2]{A=1} 6 = B + 5C \xRightarrow{B=C} 6 = 6B \Rightarrow \boxed{B=1}, \boxed{C=1}$

· 1 pt for each constant found correctly. = 4 points

$$\int \frac{2x^3-4x^2-2}{(x^2+1)(x-1)^2}\,dx = \int \left[\frac{x+1}{x^2+1} + \frac{1}{x-1} - \frac{2}{(x-1)^2}\right]dx$$

$$= \int \left[\frac{x}{x^2+1} + \frac{1}{x^2+1} + \frac{1}{x-1} - \frac{2}{(x-1)^2}\right]dx$$

$$= \underset{1}{\frac{1}{2}\ln(x^2+1)} + \underset{1}{\tan^{-1}x} + \underset{1}{\ln|x-1|} + \underset{1}{\frac{2}{x-1}} + C$$

5. **(8 + 8 = 16 points)** Determine whether the integral is convergent or divergent. If it is convergent, find its value. **You may use comparison tests.**

a) $\int_0^2 \frac{1}{\sqrt{|x-1|}}\,dx.$

It is an improper Integral since the integrand has an infinite discontinuity at $x = 1 \in [0,2]$. So (2)

$$\int_0^2 \frac{1}{\sqrt{|x-1|}}\,dx = \int_0^1 \frac{1}{\sqrt{|x-1|}}\,dx + \int_1^2 \frac{1}{\sqrt{|x-1|}}\,dx$$

- $$\int_0^1 \frac{1}{\sqrt{|x-1|}}\,dx = \lim_{t\to 1^-} \int_0^t \frac{1}{\sqrt{1-x}}\,dx = \lim_{t\to 1^-} -2\sqrt{1-x}\Big]_0^t \quad (1)$$
$$= \lim_{t\to 1^-} -2\sqrt{1-t} + 2 = 0 + 2 = 2 \quad (1) \quad \text{Conv.}$$

- $$\int_1^2 \frac{1}{\sqrt{|x-1|}}\,dx = \lim_{t\to 1^+} \int_t^2 \frac{1}{\sqrt{x-1}}\,dx = \lim_{t\to 1^+} 2\sqrt{x-1}\Big]_t^2 \quad (1)$$
$$= \lim_{t\to 1^+} 2 - 2\sqrt{t-1} = 2 - 0 = 2 \quad (1) \quad \text{Conv.}$$

So the given integral Converges (1) and its Value is $2 + 2 = 4.$ (1)

b) $\int_{e^2}^{\infty} \frac{\ln x + \sin x}{\sqrt{x}}\,dx.$

- $x \geq e^2 \Rightarrow \ln x \geq 2$
- $\sin x \geq -1$ for all x

So $\ln x + \sin x \geq 2 - 1$
$\ln x + \sin x \geq 1$ for $x \geq e^2$

App

- $\ln x + \sin x \geq 1$ for $x \geq e^2$

(3) $\frac{\ln x + \sin x}{\sqrt{x}} \geq \frac{1}{\sqrt{x}}$ for $x \geq e^2$

(3)
- $$\int_{e^2}^{\infty} \frac{1}{\sqrt{x}}\,dx = \lim_{t\to\infty} \int_{e^2}^{t} \frac{1}{\sqrt{x}}\,dx \quad (1) \quad = \lim_{t\to\infty} 2\sqrt{x}\Big]_{e^2}^{t} = \lim_{t\to\infty} 2\sqrt{t} - 2e$$
$$= \infty \quad (1)$$

Thus $\int_e^{\infty} \frac{1}{\sqrt{x}}\,dx$ diverges (1)

- Since $\frac{\ln x + \sin x}{\sqrt{x}} \geq \frac{1}{\sqrt{x}}$ for $x \geq e^2$ and $\int_{e^2}^{\infty} \frac{1}{\sqrt{x}}\,dx$ diverges, then

$\int_{e^2}^{\infty} \frac{\ln x + \sin x}{\sqrt{x}}\,dx$ diverges (1) by the Comparison test. (1)

6. (**10 points**) Find $\int \frac{x^{-1/2}}{\sqrt[3]{x}+1}\,dx$.

$$I = \int \frac{1}{\sqrt{x}\,(\sqrt[3]{x}+1)}\,dx$$

Let $u = \sqrt[6]{x}$. Then $x = u^6$, $\sqrt{x} = u^3$, $\sqrt[3]{x} = u^2$, $dx = 6u^5\,du$ 2 1 1 1

$$I = \int \frac{1}{u^3(u^2+1)} \cdot 6u^5\,du$$

$$= 6\int \frac{u^2}{u^2+1}\,du \quad 1$$

$$= 6\int \left[1 - \frac{1}{u^2+1}\right] du \quad 2$$

$$= 6\left[u - \tan^{-1} u\right] + C \quad 1$$

$$= 6\left[\sqrt[6]{x} - \tan^{-1}(\sqrt[6]{x})\right] + C \quad 1$$

DPP

7. (**14 points**)Find $\int e^x \cos^2 x \, dx$. $= I$

$I = \frac{1}{2}\int e^x (1+\cos(2x))\,dx$ 3

$= \frac{1}{2}\left[\int e^x\,dx + \int e^x\cos(2x)\,dx\right]$

$= \frac{1}{2}\left[e^x + \int e^x\cos(2x)\,dx\right]$ 1

$\int e^x\cos(2x)\,dx$: $u = e^x$, $dv = \cos(2x)\,dx$ 1

$du = e^x\,dx$, $v = \frac{1}{2}\sin(2x)$ 1

$\int e^x\cos(2x)\,dx = \frac{1}{2}e^x\sin(2x) - \frac{1}{2}\int e^x\sin(2x)\,dx$ 1

$u = e^x$, $dv = \sin(2x)\,dx$ 1

$du = e^x\,dx$, $v = -\frac{1}{2}\cos(2x)$ 1

$= \frac{1}{2}e^x\sin(2x) - \frac{1}{2}\left[-\frac{1}{2}e^x\cos(2x) + \frac{1}{2}\int e^x\cos(2x)\,dx\right]$ 1

$= \frac{1}{2}e^x\sin(2x) + \frac{1}{4}e^x\cos(2x) - \frac{1}{4}\int e^x\cos(2x)\,dx$

$\frac{5}{4}\int e^x\cos(2x)\,dx = \frac{1}{2}e^x\sin(2x) + \frac{1}{4}e^x\cos(2x)$ 1

$\int e^x\cos(2x)\,dx = \frac{2}{5}e^x\sin(2x) + \frac{1}{5}e^x\cos(2x) + K$ 1

Now

$I = \frac{1}{2}\left[e^x + \frac{2}{5}e^x\sin(2x) + \frac{1}{5}e^x\cos(2x) + K\right]$ 2

$= \frac{1}{2}e^x + \frac{1}{5}e^x\sin(2x) + \frac{1}{10}e^x\cos(2x) + C$, $C = \frac{1}{2}K$