Chapter 3 MQR Markov Chain review

p_{ij}=P$(X_{n+1}=j|X_n=i)$ $\sum_{j=1}^{m} p_{ij}=1$ $\mathbf{P}=\begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{bmatrix}$

$\boldsymbol{\pi}_n=(\pi_{1n},\pi_{2n},\cdots,\pi_{mn})$ $\sum_{i=1}^{m}\pi_{in}=1$ $\boldsymbol{\pi}_{n+r}=\begin{cases} \boldsymbol{\pi}_n\cdot\mathbf{P}\cdot\mathbf{P}\cdot\cdots\cdot\mathbf{P}=\boldsymbol{\pi}_n\cdot\mathbf{P}^r & \text{homogeneous} \\ \boldsymbol{\pi}_n\cdot\mathbf{P}^{(0)}\cdot\mathbf{P}^{(1)}\cdot\cdots\cdot\mathbf{P}^{(r-1)} & \text{non-homogeneous}\end{cases}$

${}_rp_{ij}^{(n)}=P\left(X_{n+r}=j|X_n=i\right)$ ${}_rp_{ii}^{*(n)}\leq {}_rp_{ii}^{(n)}$.
${}_rp_{ii}^{*(n)}= {}_rp_{ii}^{(n)}$ if cannot reenter state i once left i

$\lambda_{ij}(s)$ =force of transition. $\lambda_i(s)=\sum_{j=1}^{i-1}\lambda_{ij}(s)+\sum_{j=i+1}^{m}\lambda_{ij}(s)$ =force of transitioning out.

$$\frac{d}{dr}({}_rp_{ij}^{(t)})=\sum_{k\neq j}\left({}_rp_{ik}^{(t)}\lambda_{kj}(t+r)-\ {}_rp_{ij}^{(t)}\cdot\lambda_{jk}(t+r)\right)=\sum_{k\neq j}\left({}_rp_{ik}^{(t)}\lambda_{kj}(t+r)\right)-\ {}_rp_{ij}^{(t)}\cdot\lambda_j(t+r)$$

Chapter 12 MQR Multiple Life Functions

The Joint Life and Last Survivor Statuses

$T_{xy}=\min(T_x,T_y)$ $T_{\overline{xy}}=\max(T_x,T_y)$ $f_{xy}(t)={}_tp_{xy}\mu_{x+t:y+t}$ ${}_tp_{xy}=S_{xy}(t)=\exp\left(-\int_0^t\mu_{x+r:y+r}dr\right)$

$S_{\overline{xy}}(t)={}_tp_x+{}_tp_y-{}_tp_{xy}$ $f_{\overline{xy}}(t)={}_tp_x\mu_{x+t}+{}_tp_y\mu_{y+t}-{}_tp_{xy}\mu_{x+t:y+t}$

Fundamental Symmetric Relations (from $min(a,b)+max(a.b)=a+b$)

$T_{xy}+T_{\overline{xy}}=T_x+T_y$ (Random Variable)

${}_tp_{xy}+{}_tp_{\overline{xy}}={}_tp_x+{}_tp_y$ or $S_{xy}(t)+S_{\overline{xy}}(t)=S_x(t)+S_y(t)$ (Survival Function)

${}_tq_{xy}+{}_tq_{\overline{xy}}={}_tq_x+{}_tq_y$ or $F_{xy}(t)+F_{\overline{xy}}(t)=F_x(t)+F_y(t)$ (Distribution Function)

$f_{xy}(t)+f_{\overline{xy}}(t)=f_x(t)+f_y(t)$ (Density Function)

regardless of whether T_x and T_y are independent.

Deferred probabilty for last survivor: $P(m\leq K^*_{\overline{xy}}<m+n)={}_{m|n}q_{\overline{xy}}={}_{m+n}q_{\overline{xy}}-{}_mq_{\overline{xy}}={}_{m|n}q_x+{}_{m|n}q_y-{}_{m|n}q_{xy}$

Two Independent Lifetimes $\mu_{x+t:y+t}=\mu_{x+t}+\mu_{y+t}$

${}_tp_{xy}=\Pr(T_x>t\text{ and }T_y>t)={}_tp_x\ {}_tp_y$ ${}_tq_{\overline{xy}}=\Pr(T_x\leq t\text{ and }T_y\leq t)={}_tq_x\ {}_tq_y$

Force of Mortality of the Last Survivor Status: $\mu_{\overline{x+t:y+t}}=\dfrac{{}_tp_x\mu_{x+t}+\ {}_tp_y\mu_{y+t}-\ {}_tp_{xy}\mu_{xy+t}}{{}_tp_{\overline{xy}}}$

Mean, Variance and Covariance of Two Lifetimes

$\overset{o}{e}_{xy}=E(T_{xy})=\int_0^\infty t\cdot f_{xy}(t)dt=\int_0^\infty {}_tp_{xy}\ dt$ $e_{xy}=E(K^*_{xy})=\sum_{k=1}^{\infty}{}_kp_{xy}$

$\overset{o}{e}_{\overline{xy}}=E(T_{\overline{xy}})=\int_0^\infty t\cdot f_{\overline{xy}}(t)dt=\int_0^\infty t\cdot({}_tp_x\mu_{x+t}+{}_tp_y\mu_{y+t}-{}_tp_{xy}\mu_{x+t:y+t})\ dt=\overset{o}{e}_x+\overset{o}{e}_y-\overset{o}{e}_{xy}$

$e_{\overline{xy}}=E(K^*_{\overline{xy}})=\sum_{k=1}^{\infty}{}_kp_{\overline{xy}}=e_x+e_y-e_{xy}$ $E(T^2_{xy})=\int_0^\infty t^2f_{xy}(t)dt=2\int_0^\infty t\cdot{}_tp_{xy}\ dt$ $E(T^2_{\overline{xy}})=2\int_0^\infty t\cdot{}_t\,p_{\overline{xy}}dt$

$E(T_x\cdot T_y)=\int_0^\infty\int_0^\infty t_x\cdot t_y\cdot f(t_x,t_y)dt_xdt_y$ $Cov(T_x,T_y)=E(T_x\cdot T_y)-E(T_x)\cdot E(T_y)$

$E(T_{xy})+E(T_{\overline{xy}})=E(T_x)+E(T_y)$

$Var(T_{xy})+Var(T_{\overline{xy}})=Var(T_x)+Var(T_y)-2\left[(E(T_x)-E(T_{xy}))\left(E(T_y)-E(T_{xy})\right)\right]$

$Cov(T_{xy},T_{\overline{xy}})=Cov(T_x,T_y)+[E(T_x)-E(T_{xy})]\times[E(T_y)-E(T_{xy})]\overset{\text{if }T_x\ \&\ T_y\text{ independent}}{=}\left(\overset{o}{e}_x-\overset{o}{e}_{xy}\right)\left(\overset{o}{e}_y-\overset{o}{e}_{xy}\right)$

Statuses Involving the Order of Death: **Contingent Probabilities for Independent Lives**

${}_tq_{\overset{1}{x}y}=\int_0^t\Pr\left(T_y>T_x\mid T_x=u\right)f_x(u)du=\int_0^t{}_up_y\ {}_up_x\mu_{x+u}du,$

${}_tq_{\overset{2}{x}y}=\int_0^t\Pr\left(T_x<T_y\mid T_y=u\right)f_y(u)du=\int_0^t{}_uq_x\ {}_up_y\mu_{y+u}du$ ${}_tq_{\overset{1}{x}y}+{}_tq_{x\overset{1}{y}}={}_tq_{xy}$ ${}_tq_{\overset{2}{x}y}+{}_tq_{x\overset{2}{y}}={}_tq_{\overline{xy}}$

Symmetric Relation between Joint and Last Survivor Continuous Insurance

$\overline{A}_{xy}+\overline{A}_{\overline{xy}}=\overline{A}_x+\overline{A}_y$

similar relations hold for n-year term, pure endowment, and endowment insurances.

Covariance between Joint and Last Survivor Benefits

$Cov(v^{T_{xy}},v^{T_{\overline{xy}}})=Cov(v^{T_x},v^{T_y})+\left(\overline{A}_x-\overline{A}_{xy}\right)\left(\overline{A}_y-\overline{A}_{xy}\right)$

Similar relations hold for n-year term, pure endowment, and endowment insurances.

1. **Relation between Insurances and Annuities**

$\overline{a}_{xy}=\dfrac{1-\overline{A}_{xy}}{\delta},$ $\overline{a}_{\overline{xy}}=\dfrac{1-\overline{A}_{\overline{xy}}}{\delta}$ $\ddot{a}_{xy}=\dfrac{1-A_{xy}}{d},$ $\ddot{a}_{\overline{xy}}=\dfrac{1-A_{\overline{xy}}}{d}$

Similar relations hold for n-year endowment insurances and annuities.

2. **Fully Discrete Insurances and Annuities**

$$\ddot{a}_{xy} = \sum_{k=0}^{\infty} v^k \ {}_kp_{xy} \qquad A_{xy} = E\left[v^{K^*_{xy}}\right] = \sum_{k=1}^{\infty} v^k \ {}_{k-1|}q_{xy}$$

$$\ddot{a}_{\overline{xy}} = \sum_{k=0}^{\infty} v^k \ {}_kp_{\overline{xy}} \qquad A_{\overline{xy}} = E\left[v^{K^*_{\overline{xy}}}\right] = \sum_{k=1}^{\infty} v^k \ {}_{k-1|}q_{\overline{xy}} = A_x + A_y - A_{xy}$$

3. Reversionary Annuities (payment only when one life fails until the other also fails)

Payment to (y) when (x) has failed: $a_{x|y} = \sum_{k=1}^{\infty} v^k \left({}_kq_x \cdot {}_k p_y\right) = \sum_{k=1}^{\infty} v^k \left({}_kp_y - {}_k p_{xy}\right) = a_y - a_{xy}$

n-yrs (at most) pmt to (x) when (y) has failed: $a_{y|x:\overline{n}|} = \sum_{k=1}^{n} v^k \left({}_kq_x \cdot {}_k p_y\right) = \sum_{k=1}^{n} v^k \left({}_kp_y - {}_k p_{xy}\right) = a_{y:\overline{n}|} - a_{xy:\overline{n}|}$

Continuous Payment to (y) when (x) has failed: $\overline{a}_{x|y} = \int_0^{\infty} v^k \left({}_tq_x \cdot {}_t p_y\right) dt = \int_0^{\infty} v^t \left({}_tp_y - {}_t p_{xy}\right) dt = \overline{a}_y - \overline{a}_{xy}$

$$P(a_{y|x}) = \frac{a_{y|x}}{\ddot{a}_{xy}} = \frac{a_y - a_{xy}}{\ddot{a}_{xy}}. \qquad {}_tV(a_{y|x}) = \begin{cases} a_{y+t|x+t} - P(a_{y|x}) \cdot \ddot{a}_{x+t:y+t} & \text{both survives} \\ a_{x+t} & \text{if } (x) \text{ survives and } (y) \text{ fails} \\ 0 \text{ since contract expired} & \text{if } (x) \text{ fails and } (y) \text{ survives} \end{cases}$$

$a_{\overline{xy}} = a_{x|y} + a_{y|x} + a_{xy}$

4. Contingent Insurance

$$\overline{A}_{\overset{1}{x}y} = \int_0^{\infty} v^t \cdot {}_t p_{xy}\mu_{x+t}dt \qquad \overline{A}_{\overset{2}{x}y} = \int_0^{\infty} v^t \cdot {}_tp_x\mu_{x+t}(1 - {}_tp_y)dt = \overline{A}_x - \overline{A}_{\overset{1}{x}y}.$$

$$\overline{A}_{\overset{1}{x}y} + \overline{A}_{x\overset{1}{y}} = \overline{A}_{xy} \qquad \overline{A}_{\overset{2}{x}y} + \overline{A}_{x\overset{2}{y}} = \overline{A}_{\overline{xy}}.$$

5. m-thly payable multiple life benefits

under UDD: $\ddot{a}^{(m)}_{xy} \approx \alpha(m)\ddot{a}_{xy} - \beta(m) \qquad \ddot{a}^{(m)}_{\overline{xy}} \approx \alpha(m)\ddot{a}_{\overline{xy}} - \beta(m) \qquad A^{(m)}_{xy} \approx \frac{i}{i^{(m)}}A_{xy} \qquad A^{(m)}_{\overline{xy}} \approx \frac{i}{i^{(m)}}A_{\overline{xy}}$

$$\alpha(m) = s^{(m)}_{\overline{1}|}\ddot{a}^{(m)}_{\overline{1}|} = \frac{id}{i^{(m)}d^{(m)}} \qquad \beta(m) = \frac{s^{(m)}_{\overline{1}|} - 1}{d^{(m)}} = \frac{i - i^{(m)}}{i^{(m)}d^{(m)}}$$

non-UDD (Woolhouse formula): $\ddot{a}^{(m)}_{xy} \approx \ddot{a}_{xy} - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\delta + \mu_{xy})$

$$\lim_{m\to\infty} \ddot{a}^{(m)}_{xy} = \overline{a}_{xy} \approx \ddot{a}_{xy} - \frac{1}{2} - \frac{1}{12}(\delta + \mu_{xy})$$

Premiums and Reserves

$$P_{xy} = \frac{A_{xy}}{\ddot{a}_{xy}} \qquad P_{\overline{xy}} = \frac{A_{\overline{xy}}}{\ddot{a}_{\overline{xy}}} \qquad {}_tV_{xy} = A_{x+t:y+t} - P_{xy} \cdot \ddot{a}_{x+t:y+t}$$

$${}_tV_{\overline{xy}} = \begin{cases} A_{\overline{x+t:y+t}} - P_{\overline{xy}} \cdot \ddot{a}_{\overline{x+t:y+t}} & \text{if } (x) \text{ and } (y) \text{ survives} \\ A_{x+t} - P_{\overline{xy}} \cdot \ddot{a}_{x+t:} & \text{if } (x) \text{ survives and } (y) \text{ fails} \\ A_{y+t} - P_{\overline{xy}} \cdot \ddot{a}_{y+t} & \text{if } (x) \text{ fails and } (y) \text{ survives} \end{cases}$$

Dependent Life Models - Common Shock Model

$$\mu_{x+t} = \mu^*_{x+t} + \mu^c_t \overset{\text{if constant common force}}{=} \mu^*_{x+t} + \lambda \qquad \mu_{y+t} = \mu^*_{y+t} + \mu^c_t \overset{\text{if constant common force}}{=} \mu^*_{y+t} + \lambda$$

$$\mu_{x+t:y+t} = \mu^*_{x+t} + \mu^*_{y+t} + \lambda \qquad {}_tp_{xy} = \exp\left(-\int_0^t \left[\mu^*_{x+t} + \mu^*_{y+t} + \lambda\right] dt\right) = {}_tp^*_x \cdot {}_tp^*_y \cdot e^{-\lambda t}$$

${}_tp_x = {}_tp^*_x \cdot e^{-\lambda t} \qquad {}_tp_y = {}_tp^*_y \cdot e^{-\lambda t}$ Note that $\mu_{xy+t} \neq \mu_{x+t} + \mu_{y+t}$ and ${}_tp_{xy} \neq {}_tp_x \cdot {}_tp_y$

An Exponential Common Shock Model with Constant Force of Transitions (From ACTEX MLC manual)

$T_x \sim Exp(\mu^*_x + \lambda), \qquad T_y \sim Exp(\mu^*_y + \lambda), \qquad T_{xy} \sim Exp(\mu^*_x + \mu^*_y + \lambda)$

$$\overline{A}_x = \frac{\mu^*_x + \lambda}{\mu^*_x + \lambda + \delta} \qquad \overline{A}_{xy} = \frac{\mu^*_x + \mu^*_y + \lambda}{\mu^*_x + \mu^*_y + \lambda + \delta} \qquad \overline{a}_{xy} = \frac{1}{\mu^*_x + \mu^*_y + \lambda + \delta} \qquad \overline{A}_{\overline{xy}} = \overline{A}_x + \overline{A}_y - \overline{A}_{xy}$$

Chapter 13 MQR Multiple Decrement Models: Theory

OBJECTIVES: 1.To understand the concept of a multiple decrement table

2.To understand the force of decrement

3.To construct a multiple decrement model using associated single decrements and to apply various assumptions to calculate rates for discrete jumps.

13.1 Discrete Multiple Decrement Models

$$q^{(\tau)}_x = q^{(1)}_x + q^{(2)}_x + \ldots + q^{(m)}_x = \sum_{j=1}^{m} q^{(j)}_x \quad (13.1) \qquad p^{(\tau)}_x = 1 - q^{(\tau)}_x \quad (13.2)$$

$${}_np^{(\tau)}_x = 1 - {}_nq^{(\tau)}_x \quad (13.7e) \qquad \ell^{(\tau)}_{x+n} = \ell^{(\tau)}_x \cdot {}_np^{(\tau)}_x \quad (13.7f)$$

$$\ell^{(\tau)}_x = \sum_{j=1}^{m} \ell^{(j)}_x \quad (13.6) \qquad d^{(j)}_x = \ell^{(\tau)}_x \cdot q^{(j)}_x \quad (13.7a)$$

$$d^{(\tau)}_x = \sum_{j=1}^{m} d^{(j)}_x = \ell^{(\tau)}_x \cdot q^{(\tau)}_x \quad (13.3 \ \& \ 13.7b) \qquad {}_nd^{(j)}_x = \sum_{t=0}^{n-1} d^{(j)}_{x+t} = \ell^{(\tau)}_x \cdot {}_nq^{(j)}_x \quad (13.4 \ \& \ 13.7c)$$

$${}_n d_x^{(\tau)} = \sum_{j=1}^{m} {}_n d_x^{(j)} = \ell_x^{(\tau)} \cdot {}_n q_x^{(\tau)} \qquad (13.5a \ \& \ 13.7d) \qquad {}_n q_x^{(\tau)} = {}_n d_x^{(\tau)} / \ell_x^{(\tau)} = \sum_{j=1}^{m} {}_n q_x^{(j)} \qquad (13.5b)$$

13.1.2 Random Variable Analysis

The *joint probability function* of K_x^* and J_x is $\Pr(K_x^* = k \cap J_x = j) = {}_{k-1|} q_x^{(j)} = \dfrac{d_{x+k-1}^{(j)}}{\ell_x^{(\tau)}}$ (13.8)

The *marginal probability functions* are

i) $\Pr(K_x^* = k) = \sum_{j=1}^{m} \Pr(K_x^* = k \cap J_x = j) = {}_{k-1|} q_x^{(\tau)} = \dfrac{d_{x+k-1}^{(1)} + ... + d_{x+k-1}^{(m)}}{\ell_x^{(\tau)}} = \sum_{j=1}^{m} \dfrac{d_{x+k-1}^{(j)}}{\ell_x^{(\tau)}}$ (13.9)

ii) $\Pr(J_x = j) = \sum_{k=1}^{\infty} \Pr(K_x^* = k \cap J_x = j) = \sum_{k=1}^{\infty} \dfrac{d_{x+k-1}^{(j)}}{\ell_x^{(\tau)}}$ (13.10)

13.2 Theory of Competing Risks $\quad {}_n q_x'^{(j)} \geq {}_n q_x^{(j)}$ (13.11)

13.3 Continuous Multiple Decrement Models

$\mu_{x+t}^{(j)} = \dfrac{-\frac{d}{dt}\ {}_t p_x'^{(j)}}{{}_t p_x'^{(j)}}$ (13.12a) $\quad \mu_{x+t}^{(\tau)} = \dfrac{-\frac{d}{dt}\ {}_t p_x^{(\tau)}}{{}_t p_x^{(\tau)}}$ (13.13a) $\quad {}_t p_x'^{(j)} = \exp\left(-\int_0^t \mu_{x+s}^{(j)}\, ds\right)$ (13.12b)

${}_t q_x'^{(j)} = \int_0^t {}_s p_x'^{(j)} \mu_{x+s}^{(j)}\, ds = 1 - {}_t p_x'^{(j)}$ $\quad$ **Survival Probability** ${}_t p_x^{(\tau)} = \exp\left(-\int_0^t \mu_{x+s}^{(\tau)}\, ds\right)$ (13.13b)

$f_{x^{(j)}}(t) = {}_t p_x'^{(j)} \cdot \mu_{x+t}^{(j)}$ (13.14) $\quad F_{x^{(j)}}(t) = \Pr\left[T_x^{(j)} \leq t\right] = \int_0^t f_{x^{(j)}}(s)\, ds = \int_0^t {}_s p_x'^{(j)} \cdot \mu_{x+s}^{(j)}\, ds$ (13.15)

$$\begin{aligned} \mu_{x+t}^{(\tau)} &= \frac{-\frac{d}{dt}\ {}_t p_x^{(\tau)}}{{}_t p_x^{(\tau)}} = -\frac{d}{dt} \ln_t p_x^{(\tau)} = -\frac{d}{dt} \ln\left[{}_t p_x'^{(1)} \cdot\ {}_t p_x'^{(2)} \cdot ... \cdot\ {}_t p_x'^{(m)}\right] \\ &= \left(-\frac{d}{dt} \ln\ {}_t p_x'^{(1)}\right) + \left(-\frac{d}{dt} \ln\ {}_t p_x'^{(2)}\right) + ... + \left(-\frac{d}{dt} \ln\ {}_t p_x'^{(m)}\right) = \mu_{x+t}^{(1)} + \mu_{x+t}^{(2)} + ... + \mu_{x+t}^{(m)} = \sum_{j=1}^{m} \mu_{x+t}^{(j)} \qquad (13.17) \end{aligned}$$

Fundamental Relation Between Primed and Unprimed Rates: ${}_t p_x^{(\tau)} = \exp\left(-\sum_{j=1}^{m} \int_0^t \mu_{x+s}^{(j)}\, ds\right) = \prod_{j=1}^{m} {}_t p_x'^{(j)}$ (13.16)

${}_t q_x^{(j)} = \int_0^t f_{T,J}(s,j)\, ds = \int_0^t {}_s p_x^{(\tau)} \cdot \mu_{x+s}^{(j)}\, ds$ (13.18) $\quad {}_t q_x^{(\tau)} = \int_0^t {}_s p_x^{(\tau)} \cdot \mu_{x+s}^{(\tau)}\, ds$ (13.20)

$\dfrac{d}{dt}\ {}_t q_x^{(j)} = \dfrac{d}{dt} \int_0^t {}_s p_x^{(\tau)} \cdot \mu_{x+s}^{(j)}\, ds = {}_t p_x^{(\tau)} \cdot \mu_{x+s}^{(j)} \ \rightarrow \ \mu_{x+t}^{(j)} = \dfrac{\frac{d}{dt}\ {}_t q_x^{(j)}}{{}_t p_x^{(\tau)}}$ (13.19)

Joint Distribution of T_x and J_x $\quad \Pr(t < T_x \leq t + dt \text{ and } J_x = j) \approx {}_t p_x^{(\tau)} \mu_{x+t}^{(j)} dt, \quad {}_t q_x^{(j)} = \int_0^t {}_s p_x^{(\tau)}\ \mu_{x+s}^{(j)}\, ds.$

13.4.1 Uniform Distribution of Decrements in the Multiple Decrement Context

${}_t q_x^{(j)} = t \cdot q_x^{(j)}$ (13.21) $\quad q_x^{(j)} = {}_t p_x^{(\tau)} \cdot \mu_{x+t}^{(j)}$ (13.22)

${}_t q_x^{(\tau)} = t \cdot q_x^{(\tau)}$ (13.23) $\quad {}_t p_x^{(\tau)} = 1 - t \cdot q_x^{(\tau)}$ (13.24)

$\mu_{x+t}^{(j)} = \dfrac{q_x^{(j)}}{{}_t p_x^{(\tau)}} = \dfrac{q_x^{(j)}}{1 - t \cdot q_x^{(\tau)}}$ (13.25) $\quad {}_t p_x'^{(j)} = \exp\left[\dfrac{q_x^{(j)}}{q_x^{(\tau)}} \cdot \ln\left(1 - t \cdot q_x^{(\tau)}\right)\right] = \left(1 - t \cdot q_x^{(\tau)}\right)^{q_x^{(j)}/q_x^{(\tau)}}$ (13.26)

13.4.2 Uniform Distribution in the Associated Single- Decrement Tables

${}_t q_x'^{(j)} = t \cdot q_x'^{(j)}$ (13.27) $\quad {}_t p_x'^{(j)} \cdot \mu_{x+t}^{(j)} = q_x'^{(j)}$ (13.28)

Double decrement case: $q_x^{(1)} = \int_0^1 \left(1 - t \cdot q_x'^{(2)}\right) \cdot q_x'^{(1)} dt = q_x'^{(1)} \left(1 - \frac{1}{2} \cdot q_x'^{(2)}\right)$ (13.29a)

$q_x^{(2)} = q_x'^{(2)} \left(1 - \frac{1}{2} \cdot q_x'^{(1)}\right)$ (13.29b)

Triple Decrement case: $q_x^{(1)} = q_x'^{(1)} \left[1 - \frac{1}{2}\left(q_x'^{(2)} + q_x'^{(3)}\right) + \frac{1}{3}\left(q_x'^{(2)} \cdot q_x'^{(3)}\right)\right]$ (13.30)

Miscellaneous Results (From ACTEX MLC manual)

1. Assumptions on the single decrement table.

Backing out the Unprimed Rates from Primed Rates

$${}_s q_x^{(i)} = \int_0^s {}_t p_x^{(\tau)} \mu_{x+t}^{(i)}\, dt = \int_0^s \left[\prod_{j=1,\ j \neq i}^{m} {}_t p_x'^{(j)}\right] {}_t p_x'^{(i)} \mu_{x+t}^{(i)}\, dt$$

2. **Constant Force Assumption for Multiple Decrements** $\quad$ For any $t \in [0,1]$ and integer-valued x,

(i) ${}_tp_x^{(\tau)} = \left[p_x^{(\tau)}\right]^t$ (survival probability for **fractional** ages)

(ii) *Ratio* Property : $\dfrac{{}_tq_x^{(i)}}{{}_tq_x^{(\tau)}} = \dfrac{\mu_{x+s}^{(i)}}{\mu_{x+s}^{(\tau)}}$ for any $s \in [0,1]$ (To get unprimed rates from (i))

(iii) *Partition* Property : ${}_tp_x^{t(i)} = \left[{}_tp_x^{(\tau)}\right]^{q_x^{(i)}/q_x^{(\tau)}}$ (To get primed rates from unprimed rates from (i))

3. **Uniform Distribution of Death (UDD) for Multiple Decrement (MUDD) Table**
For any $t \in [0,1]$ and integer-valued x,

(i) ${}_tp_x^{(\tau)}\mu_{x+t}^{(i)} = q_x^{(i)}$ or equivalently $\mu_{x+t}^{(i)} = \dfrac{q_x^{(i)}}{1 - tq_x^{(\tau)}}$ for $t \neq 1$

(ii) *Ratio* Property : $\dfrac{{}_tq_x^{(i)}}{{}_tq_x^{(\tau)}} = \dfrac{\mu_{x+s}^{(i)}}{\mu_{x+s}^{(\tau)}}$ for any $s \in [0,1]$

(iii) *Partition* Property : ${}_tp_x'^{(i)} = \left[{}_tp_x^{(\tau)}\right]^{q_x^{(i)}/q_x^{(\tau)}}$ (To get primed rates from unprimed rates ${}_tq_x^{(i)}$ and ${}_tp_x^{(\tau)}$)

Discrete jumps: Handling Both Discrete and Continuous Decrement

1) ${}_sq_x^{(i)} = \int_0^s \left[\prod_{j=1,\ j\neq i}^{m} {}_tp_x'^{(j)}\right] {}_tp_x'^{(i)}\mu_{x+t}^{(i)}dt$ holds when decrement i is **continuous.**

2) ${}_sq_x^{(i)} = \sum_{t_k \leq\ s}\left[\prod_{j=1,\ j\neq i}^{m} {}_{t_k}p_x'^{(j)}\right]\Delta({}_{t_k}q_x'^{(i)})$ holds when decrement i is **discrete**

. Here t_k are the jump times and $\Delta\left({}_{t_k}q_x'^{(i)}\right)$ is the jump size at time t_k.

Chapter 14 MQR Multiple Decrement Models: (Applications)

14.1 Actuarial Present Value

$A_x = \sum_{k=1}^{\infty} v^k \cdot \Pr(K_x^* = k)$ (14.1) $A_x^{(j)} = \sum_{k=1}^{\infty} v^k \cdot \Pr(K_x^* = k \cap J_x = j)$ (14.2)

If the **time** and **cause** of decrement are **independent**,

$A_x^{(j)} = \sum_{k=1}^{\infty} v^k \cdot \Pr(K_x^* = k) \cdot \Pr(J_x = j)$ (14.3a) or $A_x^{(j)} = \sum_{k=1}^{\infty} v^k \cdot {}_{k-1}p_x^{(\tau)} \cdot q_{x+k-1}^{(j)}$ (14.3b)

For benefit paid at the **instant of failure** $\overline{A}_x^{(j)} = \int_0^{\infty} v^t \cdot {}_tp_x^{(\tau)} \cdot \mu_{x+t}^{(j)}\ dt$ (14.4)

14.2 Asset Shares $[{}_0AS + G(1-r_1) - e_1](1+i) = b_1^{(1)} \cdot q_x^{(1)} + b_1^{(2)} \cdot q_x^{(2)} + {}_1AS \cdot p_x^{(\tau)}$, (14.5a)

so ${}_1AS = \dfrac{[{}_0AS + G(1-r_1) - e_1](1+i) - b_1^{(1)} \cdot q_x^{(1)} - b_1^{(2)} \cdot q_x^{(2)}}{p_x^{(\tau)}}$. (14.5b)

In general, $[{}_{k-1}AS + G(1-r_k) - e_k](1+i) = b_k^{(1)} \cdot q_{x+k-1}^{(1)} + b_k^{(2)} \cdot q_{x+k-1}^{(2)} + {}_kAS \cdot p_{x+k-1}^{(\tau)}$, (14.6a)

so ${}_kAS = \dfrac{[{}_{k-1}AS + G(1-r_k) - e_k](1+i) - b_k^{(1)} \cdot q_{x+k-1}^{(1)} - b_k^{(2)} \cdot q_{x+k-1}^{(2)}}{p_{x+k-1}^{(\tau)}}$. (14.6b) $U_k = {}_kAS - {}_kV^G$. (14.7)

14.3 Non-Forfeiture Options

14.3.1 Cash Value ${}_tCV_x$

14.3.2 Reduced Paid-up Insurance

$RPU = \dfrac{{}_tCV_x}{A_{x+t}}$, (14.8) ${}_tW_x = \dfrac{{}_tV_x}{A_{x+t}}$, (14.9)

14.3.3 Extended Term Insurance ${}_tCV_x = A^1_{x+t:n\rceil}$ (14.10) ${}_tCV_{x:n\rceil} = A^1_{x+t:n-t\rceil} + PE \cdot {}_{n-t}E_{x+t}$, (14.11)

14.4 Multi State Model Representation

14.4.2 The Total and Permanent Disability Model

${}^h\overline{A}_x^{(f)} = \int_0^{\infty} v^r \cdot {}_rp_x^{(\tau)} \cdot \mu_{x+r}^{(j)}\ dr$ (14.12a) ${}^h\overline{A}_x^{(f)} = \int_0^{\infty} v^r \cdot {}_rp_{11}^{(0)} \cdot \lambda_{13}(r)\ dr$ (14.12b)

${}^d\overline{A}_{x+r} = \int_0^{\infty} v^s \cdot {}_sp_{x+r}^d \cdot \mu_{x+r+s}^d\ ds$ (14.13a) ${}^d\overline{A}_{x+r} = \int_0^{\infty} v^s \cdot {}_sp_{22}^{(r)} \cdot \lambda_{23}(s)\ ds$ (14.13b)

${}^h\overline{A}_x^d = \int_0^{\infty} v^r \cdot {}_r\,p_x^{(\tau)} \cdot \mu_{x+r}^{(d)} \cdot {}^d\,\overline{A}_{x+r}dr = \int_0^{\infty} v^r \cdot {}_r\,p_x^{(\tau)} \cdot \mu_{x+r}^{(d)}\left(\int_0^{\infty} v^s \cdot {}_s\,p_{x+r}^d \cdot \mu_{x+r+s}^d\ ds\right)dr$ (14.14a)

${}^h\overline{A}_x^d = \int_0^{\infty} v^r \cdot {}_rp_{11}^{(0)} \cdot \lambda_{12}(r)\left(\int_0^{\infty} v^s \cdot {}_s\,p_{22}^{(r)} \cdot \lambda_{23}(s)\ ds\right)dr$, (14.14b)

${}^h\overline{a}_x^d = \int_0^{\infty} v^r \cdot {}_r\,p_x^{(\tau)} \cdot \mu_{x+r}^{(d)} \cdot {}^d\,\overline{a}_{x+r}\ dr = \int_0^{\infty} v^r \cdot {}_r\,p_x^{(\tau)} \cdot \mu_{x+r}^{(d)}\left(\int_0^{\infty} v^s \cdot {}_s\,p_{x+r}^d\ ds\right)dr$ (14.15a)

${}^h\overline{a}_x^d = \int_0^{\infty} v^r \cdot {}_rp_{11}^{(0)} \cdot \lambda_{12}(r)\left(\int_0^{\infty} v^s \cdot {}_s\,p_{22}^{(r)}\ ds\right)dr$, (14.15b)

14.4.3 Disability Model Allowing For Recovery $f(x+\Delta x) \approx f(x) + f'(x)\cdot \Delta x$ (14.16)

$$\begin{aligned}\frac{d}{dr}\ {}_rp_{11}^{(t)} &= {}_rp_{12}^{(t)}\cdot \lambda_{12}(t+r) - {}_rp_{11}^{(t)}\cdot \lambda_{12}(t+r) \quad \text{(at } k\text{=2)}\\ &\quad + {}_rp_{13}^{(t)}\cdot \lambda_{31}(t+r) - {}_rp_{11}^{(t)}\cdot \lambda_{13}(t+r) \quad \text{(at } k \text{ of 3)}\\ &= {}_rp_{12}^{(t)}\cdot \lambda_{12}(t+r) - {}_rp_{11}^{(t)}\cdot [\lambda_{12}(t+r)+\lambda_{13}(t+r)], \qquad (14.17)\end{aligned}$$

$$\begin{aligned}\frac{d}{dr}\ {}_rp_{12}^{(t)} &= {}_rp_{11}^{(t)}\cdot \lambda_{12}(t+r) - {}_rp_{12}^{(t)}\cdot \lambda_{21}(t+r) \quad \text{(at } k\text{=1)}\\ &\quad + {}_rp_{13}^{(t)}\cdot \lambda_{32}(t+r) - {}_rp_{12}^{(t)}\cdot \lambda_{23}(t+r) \quad \text{(at } k \text{ of 3)}\\ &= {}_rp_{11}^{(t)}\cdot \lambda_{12}(t+r) - {}_rp_{12}^{(t)}\cdot [\lambda_{21}(t+r)+\lambda_{23}(t+r)] \qquad (14.18)\end{aligned}$$

$${}_{r+\Delta r}p_{ij}^{(0)} \approx {}_rp_{ij}^{(0)} + \frac{d}{dr}\ {}_rp_{ij}^{(0)}\cdot \Delta r \qquad (14.19)$$

$${}_{r+\Delta r}p_{11}^{(0)} \approx {}_rp_{11}^{(0)} + \Delta r\left\{{}_rp_{12}^{(0)}\cdot \lambda_{21}(r) - {}_rp_{11}^{(0)}\cdot[\lambda_{12}(r)+\lambda_{13}(r)]\right\}, \qquad (14.20)$$

$${}_{r+\Delta r}p_{12}^{(0)} \approx {}_rp_{12}^{(0)} + \Delta r\left\{{}_rp_{11}^{(0)}\cdot \lambda_{12}(r) - {}_rp_{12}^{(0)}\cdot[\lambda_{21}(r)+\lambda_{23}(r)]\right\}, \qquad (14.21)$$

14.4.5 Thiele's Differential Equation in the Multiple Decrement Case

$\bar{a}_x^{(\tau)} = \int_0^\infty v^r\cdot\ {}_rp_x^{(\tau)} dr$ (14.22) $\bar{P} = \dfrac{APVB}{\bar{a}_x^{(\tau)}} = \dfrac{APV_x^T}{\bar{a}_x^{(\tau)}} = \dfrac{\sum_{j=1}^m APV_x^{(j)}}{\bar{a}_x^{(\tau)}}$ (14.23)

${}^d\bar{a}_{x+t} = \int_0^\infty v^s\cdot\ {}_sp_{x+t}^d\ dr$ (14.24) $\dfrac{d}{dt}\ {}_t^h\overline{V} = \bar{P} + \delta\cdot {}_t^h\overline{V} - \mu_{x+t}^{(f)}\left(1 - {}_t^h\overline{V}\right) - \mu_{x+t}^{(d)}\left({}_t^d\overline{V} - {}_t^h\overline{V}\right).$ (14.25)

$$\frac{d}{dt}\ {}_t^d\overline{V} = \delta\cdot {}_t^d\overline{V} - 1 - {}^d\mu_{x+t}^{(f)}\left(1 - {}_t^d\overline{V}\right) - {}^d\mu_{x+t}^{(r)}\left({}_t^h\overline{V} - {}_t^d\overline{V}\right) \qquad (14.26)$$

$$\frac{{}_t^h\overline{V} - {}_{t-\Delta t}^{\ h}\overline{V}}{\Delta t} \approx \bar{P} + \delta\cdot {}_t^h\overline{V} - \mu_{x+t}^{(f)}\left(1 - {}_t^h\overline{V}\right) - \mu_{x+t}^d\left({}_t^d\overline{V} - {}_t^h\overline{V}\right),$$

or $${}_{t-\Delta t}^{\ h}\overline{V} \approx {}_t^h\overline{V} - \Delta t\left\{\bar{P} + \delta\cdot {}_t^h\overline{V} - \mu_{x+t}^{(f)}\left(1 - {}_t^h\overline{V}\right) - \mu_{x+t}^d\left({}_t^d\overline{V} - {}_t^h\overline{V}\right)\right\} \qquad (14.27)$$

$${}_{t-\Delta t}^{\ d}\overline{V} \approx {}_t^d\overline{V} - \Delta t\left\{\delta\cdot {}_t^d\overline{V} - 1 - {}^d\mu_{x+t}^{(f)}\left(1 - {}_t^d\overline{V}\right) - {}^d\mu_{x+t}^{(r)}\left({}_t^h\overline{V} - {}_t^d\overline{V}\right)\right\} \qquad (14.28)$$

14.5 Defined Benefit (DB) Pension Plans

14.5.1 Normal Retirement (NR) Benefits

Projected Annual Benefit $PAB_z = 0.01p\cdot YOS_z\cdot FAS_z$ (14.29)

Final Annual Salary $FAS_z = \dfrac{1}{3}\left(\dfrac{S_{z-3}+S_{z-2}+S_{z-1}}{S_x}\right)\cdot CAS_x$ (14.30)

Projected Aggregate Salary $PAS_z = \dfrac{1}{S_x}\sum\limits_{k=x}^{z-1} S_k\cdot CAS_x$ (14.31)

Projected Annual retirement Benefit: $PAB_z = 0.01p\cdot PAS_z$ (14.32)

APV of the projected benefit, at age x: $APV_x^{NR} = PAB_z\cdot v^{z-x}\cdot\ {}_{z-x}p_x^{(\tau)}\cdot\ {}^r\ddot{a}_z^{(12)}.$ (14.33)

14.5.2 Early Retirement (ER) Benefits

$$APV_{35}^{ER} = \sum_{y=60}^{64} PAB_{y+1/2}\cdot\left[1 - 0.05\left(65 - y - \frac{1}{2}\right)\right]\cdot v^{y+1/2-35}\cdot\ {}_{y-35}p_{35}^{(\tau)}\cdot q_y^{(r)}\cdot\ {}^r\ddot{a}_{y+1/2}^{(12)} \qquad (14.34)$$

14.5.3 Withdrawal and other Benefits

Assuming a 5-year vesting rule and assuming employees take their *withdrawal* benefit at NRA, the APV at age 35 is

$$APV_{35}^W = \sum_{y=35+5}^{59} PAB_{y+1/2}\cdot v^{30}\cdot\ {}_{y-35}p_{35}^{(\tau)}\cdot q_y^{(w)}\cdot\ {}^{\ w}_{65-y-1/2}p_{y+1/2}\cdot\ {}^r\ddot{a}_{65}^{(12)} \qquad (14.35)$$

14.5.4 Funding and Reserving

Normal Cost (Early Age) $NC_x^{EAN} = \dfrac{APV_x^T}{\ddot{a}_{x:\overline{z-x}|}^{(\tau)}}$ (14.36)

${}_tV_x^T = APV_{x+t}^T - NC_x^{EAN}\cdot \ddot{a}_{x+t:\,\overline{z-x-t}|}^{(\tau)}$ (14.37a) or retrospectively as ${}_tV_x^T = NC_x^{EAN}\cdot \ddot{s}_{x:\,\overline{t}|}^{(\tau)}$ (14.37b)

APV of the benefit accrued between ages x and $x+1$: $APV_x^{NR} = (AB_{x+1} - AB_x)\cdot v^{z-x}\cdot\ {}_{z-x}p_x^{(\tau)}\cdot\ {}^r\ddot{a}_z^{(12)}$ (14.38)

14.5 Gain and Loss Analysis

Profit with all anticipated factors:

$$P(0) = [{}_tV + G(1-r_{t+1}) - e_{t+1}](1+i_{t+1}) - \left[\left(b_{t+1}^{(1)} + s_{t+1}^{(1)}\right)\cdot q_{x+t}^{(1)}\left(b_{t+1}^{(2)} + s_{t+1}^{(2)}\right)\cdot q_{x+t}^{(2)} + p_{x+t}^{(\tau)}\cdot\ {}_{t+1}V\right] \qquad (14.39)$$

Profit with some actual experience in place of anticipated factors:

$P(1) = (14.39)$ with all anticipated factors except actual value for 1 factor.
$P(2) = (14.39)$ with all anticipated factors except actual value for 2 factors
$P(3) = (14.39)$ with all anticipated factors except actual value for 3 factors.
$P(4) = (14.39)$ with all anticipated factors except actual value for 4 factors.
Gain from factor whose gain is calculated first is $G^{F_1} = P(1) - P(0)$ (14.40a)
Gain from factor whose gain is calculated second is $G^{F_2} = P(2) - P(1)$ (14.40b)
Gain from factor whose gain is calculated third is $G^{F_3} = P(3) - P(2)$ (14.40c)
Gain from factor whose gain is calculated fourth is $G^{F_4} = P(4) - P(3)$ (14.40d)
Total gain $G^T = G^{F_1} + G^{F_2} + G^{F_3} + G^{F_4} = P(4) - P(0)$ (14.41)
When death occurs throughout year but withdrawal only at end of year, the anticipated profit expression is
$$P(0) = \left[{}_tV + G\left(1 - r_{t+1}\right) - e_{t+1}\right]\left(1 + i_{t+1}\right) - \left[\left(b_{t+1}^{(1)} + s_{t+1}^{(1)}\right) \cdot q_{x+t}^{\prime(1)}\left(b_{t+1}^{(2)} + s_{t+1}^{(2)}\right)\left(1 - q_{x+t}^{\prime(1)}\right) \cdot q_{x+t}^{\prime(2)} + p_{x+t}^{(\tau)} \cdot {}_{t+1}V\right] \quad (14.42$$

ACTEX MLC Chapter 9 Study Manual Vol II Multiple Decrement Models:Applications

Thiele's Differential Equation under Multiple Decrement
$$\frac{d\ {}_tV^g}{dt} = G_t(1 - c_t) - e_t + \left(\delta + \mu_{x+t}^{(\tau)}\right){}_tV^g - \sum_{j=1}^{n}\left(b_t^{(j)} + E_t^{(j)}\right)\mu_t^{(j)}$$
Recursive Relation for Expected Asset Shares
$$\left[{}_hAS + G_h\left(1 - c_h\right) - e_h\right](1 + i) = p_{x+h}^{(\tau)}\ {}_{h+1}AS + q_{x+h}^{(1)}\ {}_{h+1}CV + q_{x+h}^{(2)}\ b_{h+1}$$

Chapter 15 MQR Models with Variable Interest Rates

15.4 Forward Interest Rates $\left(1 + y_5\right)^5 = \left(1 + y_1\right)^1 \cdot \left(1 + f_{1,4}\right)^4.$ (15.1)
$\left(1 + y_4\right)^4 = \left(1 + y_2\right)^2 \cdot \left(1 + f_{2,2}\right)^2$ (15.2) $\left(1 + y_k\right)^k \cdot \left(1 + f_{k,5-k}\right)^{5-k} = \left(1 + y_5\right)^{k+5-k} = \left(1 + y_5\right)^5.$ (15.3)
$\left(1 + y_2\right)^2 = \left(1 + f_{1,1}\right)\left(1 + f_{0,1}\right)$

Chapter 12 ACTEX MLC Study Manual Vol II Interest Rate Risk

Spot interest rate $v(t) = \left(1 + y_t\right)^{-t}$ **Forward interest rate** $\left(1 + f_{t,k}\right)^k = \dfrac{\left(1 + y_{t+k}\right)^{t+k}}{\left(1 + y_t\right)^t} = \dfrac{v(t)}{v(t+k)}$

Chapter 16 MQR Universal Life Insurance

16.2 Indexed Universal Life Insurance.

a) Point-to-point method: $i_P = \dfrac{\textbf{Final}\text{ Index Closing Value}}{\textbf{Initial}\text{ Index Closing Value}} - 1,$ (16.1)

b) Monthly average method: $i_{MA} = \dfrac{\frac{1}{12}\sum \textbf{Monthly}\text{ Index Closing Values}}{\textbf{Initial}\text{ Index Closing Value}} - 1.$ (16.2)

16.3 Pricing Considerations

Mortality rate, Lapse rate, Expenses, Investment Income.

Double decrement model: $p_x^{(\tau)} = 1 - q_x^{(\tau)} = 1 - q_x^{(d)} - q_x^{(w)}.$ (16.3)

Withdrawal at end of year only: $p_x^{(\tau)} = \left(1 - q_x^{(d)}\right)\left(1 - q_x^{(w)}\right).$ (16.4)

Pricing for Secondary Guarantees: a) Stipulated premium method, b) Shadow fund method.

16.4 Reserving Considerations

ULI	**Universal Life Insurance**. Policy is marked by (a) extensive policyholder *choice*, (b) policyholder *participation* in interest rate risk, and (c) *secondary guarantee* features of coverage
VUL	**Variable Universal Life** insurance. *Separate investment accounts* for net contributions.
$EIUL$	**Equity-Indexed Universal Life** insurance. Interest/investment is credited to contract at rate that depends on some *published stock index* such as SP500, DJIA, or EAFE index
SC_t	**Surrender Charge** at time t.
$M\&E_t$	**Mortality and Expenses** Charge at time t.
NAR_t	**Net Amount at Risk** at time t.
AV_t	**Account Value** at time t.
CV_t	**Cash Value** at time t. ($CV_t = AV_t - SC_t$)
$NAIC$	National Association of Insurance Commissioners
PG	Policy Guarantees (Guarantees given as part of an insurance policy).
GMP	**Guaranteed Maturity Premium**. Level gross premium sufficient to endow the policy at its maturity date based on the policy guarantees of premium loads, interest rates, and expense and mortality charges.
GMF	**Guaranteed Maturity Funds**. Calculated based on the roll forward of the GMP and the policy guarantees.
GDB	Guaranteed Death Benefits.

GMB	Guaranteed Maturity Benefits.
$PVFB_t$	**Present Value** at time t of the projected **Future Benefits**.
$PVFP_t$	**Present Value** at time t of the Future **GMP stream**.
$CRVM$	Commissioner's reserve valuation method
CSV_t	**Cash Surrender Value** at time t.
AMR	**Alternative Minimum Reserves**.

Roll Forward =bring a financial value forward to the future .

16.4.1 Basic Universal Life (ULI)

Process for 1983 **NAIC** regulation to define a *minimum reserving standard* for UL products.

a) At policy issue,

1. a guaranteed maturity premium (GMP) is calculated as the level gross premium sufficient to endow the policy at its maturity date. The GMP is based on the **policy guarantees** of premium loads, interest rates, and expense and mortality charges.

   GMP_0 is policy guarantees of f(premium loads, i, $M\&E$).

2. a sequence of **guaranteed maturity funds** (GMF) is calculated based on the roll forward of the GMP and the policy guarantees

   a sequence GMF=roll forward of $f(GMP$, policy guarantees).

b) At the valuation date, t,

3. actual AV_t determined by the account value roll forward process.

4. the ratio of the actual account value to the GMF is calculated as $r_t = \dfrac{AV_t}{GMF_t}$, $\quad r_t \le 1 \qquad (16.5)$

5. $\max(AV_t, GMF_t)$ is projected forward based on the GMP and the policy guarantees. This produces a sequence of GDB and GMB.

6. $PVFB_t$ and $PVFP_t$ are calculated using valuation assumptions. Then the pre-floor CRVM reserve is defined as

   ${}_tV^{pre-floor\ CRVM} = r_t(PVFB_t - PVFP_t) \qquad (16.6)$ with r_t as defined above.

7. ${}_tV^{floor\ CRVM} = max(\frac{1}{2}$-month term reserve based on minimum valuation mortality and interest, $CSV_t)$.

8. ${}_tV^{final\ CRVM} = max(\ {}_tV^{pre-floor\ CRVM},\ {}_tV^{floor\ CRVM})$.

The regulation also defines **alternative minimum reserves** (AMR).

1. the valuation net premium is calculated at policy issue ($t = 0$) based on the GMP and the policy guarantees.

2. If the GMP $<$ the valuation net premium (VNP),

   the reserve held $=\max(a, b)$

   where a =the reserve calculated using the **actual method** and **assumptions** of the policy + VNP,

   b= the reserve calculated using the **actual method** but with **minimum valuation** assumptions + GMP).

16.4.1 Indexed Universal Life (eIUL)

NAIC Actuarial Guideline 36 (AG 36) specifies the valuation standards for IUL contracts. 3 computational methods:

1) The **implied guaranteed rate** (IGR) method: which requires insurers to satisfy the hedged-as-required criteria.These criteria set forth a strenuous constraint requiring exact, or nearly exact, hedging, as well as an indexed interest-crediting term of *not more than one year*.

2) The **CRVM** with **updated market value** (CRVM/UMV) method:

must be used if the contract has an *indexed interest-crediting term of more than one year*, or if the renewal participation rate guarantee gives an implied guaranteed rate greater than the maximum valuation rate. This method can be volatile when market conditions change.

(3) The **CRVM** with **updated average market value** (CRVM/UAMV) method:

is a hybrid of the other two, designed for an insurer who **qualifies for the first method** above but **does not wish to satisfy** the *hedged-as-required* criteria.

The CRVM/UMV method has calculation steps as follows:

a) The **issue date** ($t = 0$) calculations are as follows:

1. An **implied guaranteed interest rate** (IGR) for the duration of the initial term,

   is the **guaranteed rate** plus the **accumulated option cost** expressed as a percentage of the policy value to which the indexed benefit is applied. In turn, the **accumulated option cost** is the amount needed to provide the index-based benefit in excess of any other interest rate guarantee, accumulated to the end of the initial term at the appropriate maximum valuation rate.

2. An **implied guaranteed rate** for the period after the initial term.

3. The GMP, GMF, and valuation net premium based on the implied guaranteed rate.

b) The **valuation date** ($t = t$) calculations are as follows:

1. The implied guaranteed rate for the remainder of the current period, using the **option cost** based on the market conditions at the valuation date.

2. The implied guaranteed rate for the period following the current period, based on the option cost on the valuation date.

3. A **re-projection of future guaranteed benefits** based on the implied valuation date.

4. The **present value** of the re-projected future guaranteed benefits.

Note that the GMP, GMF, and valuation net premium remain the same as calculated at issue ($t = 0$).

16.4.4 Contracts with Secondary Guarantees

NAIC Actuarial Guideline 38 (AG 38) for reserves for UL products with secondary guarantees have 9 steps as follows:

1. The **minimum gross premium** required to satisfy the secondary guarantees is derived at issue ($t = 0$) of the contract; the value of this premium will depend on whether the *stipulated premium* or the *shadow fund* method is in use. Its calculation uses the *policy charges* and *credited interest rate* guaranteed in the contract.

2. The **basic and deficiency reserves** for the secondary guarantees are calculated using the *minimum gross premium* described in Step (1).

3. The amount of **actual contributions** made in **excess of the minimum gross premiums** is determined, again with the process depending on whether the *stipulated premium* method or the *shadow fund* method is used.

4. At the valuation date, t, a determination is made regarding **amounts needed to fully fund** the secondary guarantee.

   (a) Under the **shadow fund** method, this would be the *amount of the shadow fund account* needed to fully fund the guarantee.

   (b) Under contracts not using the shadow fund method, this would be the amount of cumulative premiums paid in excess of the required level such that no future premiums are required to fully fund the guarantee.

   Special rules apply to policies for which the secondary guarantee cannot be fully funded in advance. Here a prefunding ratio, r, ($r \leq 1$), is calculated that measures the level of prefunding for the secondary guarantee, and is eventually used in the calculation of reserves. It is defined as

   $$r = \frac{\textit{Excess Payment}}{\textit{Net Single Premium Required to Fully Fund the Guarantee}}. \qquad (16.7)$$

5. At the valuation date, t, the **net single premium** for the secondary guarantee coverage for the remainder of the secondary guarantee period is computed. NSP_t.

6. A net amount of **additional premiums** is determined by multiplying the prefunding ratio described in Step (4) times the difference between the net single premium of Step (5) and the basic plus deficiency (if any) reserve of Step (2). $r(NSP_t - {}_{bpd}V)$

7. A **reduced deficiency reserve** is determined by multiplying the deficiency reserve (if any) by the complement of the pre-funding ratio from Step (4). $(1 - r)_d V$

8. Then the **actual reserve** is the lesser of (a) the net single premium of Step (5), or (b) the amount in Step (6) plus the basic and deficiency (if any) reserve from Step (2). This result might be reduced by applicable policy surrender charges. $V = min(NSP_t, Step6 + Step2)$

9. An **increased basic reserve** is computed by subtracting the reduced deficiency reserve of Step (7) from the reserve computed in Step (8), which then becomes the basic reserve. ${}^bV = Step8 - Step7.$

ACTEX MLC Chapter 14 Study Manual Vol II Universal Life Insurance

14.1 Basic Policy Design

Account Value Accumulation $AV_t = (AV_{t-1} + P_t - EC_t - CoI_t)\,(1 + i_t^c)$

14.2 Cost of Insurance and Surrender Value

Total Death Benefit

Specified Amount (Type A): max $(FA, \gamma_t\, AV_t)$

Specified Amount plus the Account Value (Type B): max $(AV_t + X, \gamma_t\, AV_t)$

Additional Death Benefit

Specified Amount (Type A): $ADB_t = \max\,(FA - AV_t, (\gamma_t - 1)\,AV_t)$

Specified Amount plus the Account Value (Type B): $ADB_t = \max\,(X, (\gamma_t - 1)\,AV_t)$

General Formula for the cost of Insurance $CoI_t = q_t^* \times v_q \times ADB_t$

where CoI_t is the **cost of insurance** for the t^{th} time period, deducted from the account value at time $t-1$,

q_t^* is the death probability (for the t^{th} time period) used to calculate the cost of insurance,

v_q is the discount factor for discounting the cost of insurance to time $t-1$,

ADB_t is the additional death benefit at time t.

Cost of Insurance for a Specified Amount (Type A) Policy

$$CoI_t = \max\left(CoI_t^f, CoI_t^c\right)$$

where $$CoI_t^f = \frac{q_t^* v_q\,(FA - (AV_{t-1} + P - EC_t)\,(1 + i_t^c))}{1 - q_t^* v_q\,(1 + i_t^c)}$$

$$CoI = \frac{q_t^* v_q\,(1 + i_t^c)\,(\gamma_t - 1)\,(AV_{t-1} + P - EC_t)}{1 + q_t^* v_q\,(1 + i_t^c)\,(\gamma_t - 1)}$$

Cost of Insurance for a Specified Amount plus the Account Value (Type B) Policy

$$CoI_t = \max\left(CoI_t^f, CoI_t^c\right)$$

where $CoI_t^f = q_t^* v_q X$ and $$CoI_t^c = \frac{q_t^* v_q\,(1 + i_t^c)\,(\gamma_t - 1)\,(AV_{t-1} + P - EC_t)}{1 + q_t^* v_q\,(1 + i_t^c)\,(\gamma_t - 1)}$$

14.5 Profit Testing

Expected Profit: $Pr_t = (AV_{t-1} + P - EC_t)\,(1 + i_t^e) - EDB_t - ESB_t - EAV_t$

Chapter 17 MQR Deferred Variable Annuities

17.2 Deferred Annuity Products

17.2.2 Variable Deferred Annuity

Investment Advisory Fee: $$IAF_t(n) = FV_{t-1}(n) \cdot \left[(1 + IAF_t(n))^{1/365} - 1\right]. \qquad (17.1)$$

Net Investment Rate for day t : $$NIF_t(n) = \frac{NII_t(n) - IAF_t(n) + RCG_t(n) + UCG_t(n)}{FV_{t-1}(n)}, \qquad (17.2)$$

Net Investment Factor: $NIF_t(n) = 1 + NIR_t(n).$ (17.3)

Sub-account n Fund Value recursion: $FV_t(n) = FV_{t-1}(n) \cdot NIF_t(n) - EXP_t(n),$ (17.4)

Overall contract Account Value on day t : $AV_t = \sum_n FV_t(n).$ (17.5)

17.2.3 Equity -Indexed Deferred Annuity

a) point-to-point: $$i_P = \frac{\text{Index value on \textbf{closing day} of index period}}{\text{Index value on \textbf{initial day} of index period}} - 1, \qquad (17.6)$$

b) monthly ave: $$i_{MA} = \frac{\frac{1}{12n}\,[\text{\textbf{Sum} of index values on the \textbf{last day of each month} during \textbf{index period}}]}{\text{Index value on \textbf{initial day} of index period}} - 1.$$
(17.7)

c) with ratcheting $$i_P = \frac{\text{Index value on \textbf{closing day} } t \text{ of index period}}{\text{Index value on \textbf{day} } t-1 \text{ of index period}} - 1,$$

d) with ratcheting $$i_{MA} = \frac{\frac{1}{12n}\,[\text{\textbf{Sum} of index values on the \textbf{last day of each month} during \textbf{index period}}]}{\text{Index value on \textbf{previous day} } t-1 \text{ of index period}} - 1.$$

17.3 Immediate Annuity Products

17.3.2 Variable Immediate Annuity $APU = \dfrac{P_1}{PUV_1}$. (17.8)

$PUV_t = PUV_{t-1}\left(\dfrac{NIF_t}{1+AIR}\right)$, (17.9) $P_t = (APU)\,(PUV_t)$. (17.10)

$P_t = (APU)\,(PUV_{t-1})\left(\dfrac{NIF_t}{1+AIR}\right) = P_{t-1}\left(\dfrac{NIF_t}{1+AIR}\right)$. (17.11)

ACTEX MLC Chapter 13 Profit Testing

Profit vector $\Pr = (\Pr_0, \Pr_1, \Pr_2, \Pr_3 \ldots.)'$

Profit signature $\Pi = (\Pi_0, \Pi_1, \Pi_2, \Pi_3, \ldots.)' = (\Pr_0, \Pr_1, p_x \Pr_2, {}_2p_x \Pr_3, \ldots.)'$

Expected profit that emerges in $(h+1)^{th}$ year

$\Pr_{h+1} = N[({}_hV^g + G_h(1-c_h) - e_h)\,(1+i) - (b_{h+1}+E_{h+1})q_{x+h} + p_{x+h}\ {}_{h+1}V^g]$

$= N[(G_h(1-c_h) - e_h)(1+i) - (b_{h+1}+E_{h+1})q_{x+h} + (1+i){}_hV^g - p_{x+h}\cdot\ {}_{h+1}V^g]$ (1)

$\Pr_1$ =equation (1) without acquisition cost. $\Pr_0 = -$ all acquisition costs

Extension to Multiple State Models (assuming $N=1$)

$\Pr^{(j)}_{h+1} = [{}_hV^{(j)} + G^{(j)}_h(1-c^{(j)}_h) - e^{(j)}_h](1+i) - \sum_{k=0}^{n}(b^{(jk)}_{h+1} + E^{(jk)}_{h+1} + {}_{h+1}V^{(k)})p^{jk}_{x+h}$

$\Pi = (\Pi_0, \Pi_1, \Pi_2, \ldots.)'$, $\Pi_t = \sum_{k=0}^{n} {}_{t-1}p^{0k}_x \Pr^{(j)}_t$ $\Pi_0 = \Pr^{(0)}_0$,

$\Pi_1 = \sum_{k=0}^{n} {}_0p^{0k}_x \Pr^{(j)}_t = {}_0p^{00}_x \Pr^{(0)}_1 = \Pr^{(0)}_1$

Traditional Insurance Policies with Withdrawal (assuming $N=1$)

$\Pr^{(0)}_{h+1} = [{}_vV^{(0)} + G_h(1-c_h) - e_h](1+i) - [{}_{h+1}V^{(0)}p^{(\tau)}_{x+h} + {}_{h+1}CV q^{(1)}_{x+h} + (b_{h+1}+E_{h+1})q^{(2)}_{x+h}]$ $h \geq 0$ and

$\Pr^{(1)}_{h+1} = 0$, $\Pr^{(2)}_{h+1} = 0$, $h \geq 1$ $\Pr^{(0)}_0 = -$ acquisition costs

$\Pi_h = {}_{h-1}p^{(\tau)}_x \Pr^{(0)}_h$, $h \geq 2$, $\Pi_1 = \Pr^{(0)}_1$, $\Pi_0 = \Pr^{(0)}_0$

Extension to Policies with Continuous Benefit (assuming $N=1$)

$\Pr_{h+1} = [{}_hV^g + G_h(1-c_h) - e_h](1+i) - \left[(1+i)^{1/2}(b_{h+0.5} + E_{h+0.5})q_{x+h} + p_{x+h}\ {}_{h+1}V^g\right]$.

13.2 Profit Measures

1. **Net present value (NPV):** $NPV(r) = \sum_{k=0}^{n} \dfrac{C_k}{(1+r)^k}$.

   If the $NPV > 0$, then the investment is deemed **profitable**.

2. **Internal rate of return (IRR):** The internal rate of return is the zero of the equation

$$NPV(r) = \sum_{k=0}^{n} \frac{C_k}{(1+r)^k} = 0.$$

3. **Discounted payback period (DPP)**

   Given the hurdle rate r, the **discounted payback** period (also known as the **break-even** period) is the smallest value of m such that $\sum_{k=0}^{m} \dfrac{C_k}{(1+r)^k} \geq 0$.

   DPP is the **time until the investment starts** to **make** a **profit**.

4. **Profit margin**

   Profit margin is the **NPV of the net cash flows** as a **percentage of the NPV of the revenues**. Suppose that the revenue cash flows are $R_0, R_1, R_2, \ldots R_n$. Then the **profit margin** is $\dfrac{NPV(r)}{\sum_{k=0}^{n} \dfrac{R_k}{(1+r)^k}}$

For **life insurance**, $C_k = \Pi_k$, $R_k = G_k\ {}_kp_x$, NPV is expected present value of the profits at issue and premiums as revenues (mortality rate taken into account).

$$\text{Profit margin} = \frac{NPV(r)}{\sum_{k=0}^{n} \frac{G_k\ {}_kp_x}{(1+r)^k}} = \frac{\sum_{k=0}^{n} \frac{\Pi_k}{(1+r)^k}}{\sum_{k=0}^{n} \frac{G_k\ {}_kp_x}{(1+r)^k}}$$

5. **NPV** as a **proportion** of the **acquisition costs** $= \dfrac{1}{\Pi_0} NPV(r) = \dfrac{1}{\Pi_0}\sum_{k=0}^{n} \dfrac{\Pi_k}{(1+r)^k}$.