King Fahd University of Petroleum & Minerals
Department of Math. & Stat.

Math 690 - Final Exam (131)   Time: 2 H 30 min

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Name: ID #
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Problem 1	/15
Problem 2	/5
Problem 3	/15
Problem 4	/10
Problem 5	/5
Total	/50

Problem # 1. (15 marks) Let Ω be a bounded and smooth domain of $\mathbb{R}^4$ and $f \in L^p(\Omega), p > 2$. Consider the problem

$$(P) \qquad \begin{cases} -\Delta u + u = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

i) Show that (P) has a unique weak solution $u \in H_0^1(\Omega) \cap H^2(\Omega)$.

ii) Show that $u \in W^{1,4}(\Omega)$

iii) Show that $u \in L^p(\Omega)$ and $||u||_p \leq ||f||_p$

iv) If $f \in L^\infty(\Omega)$, show that $u \in L^\infty(\Omega)$

Hint: Remember that if $v \in L^q(\Omega)$, $\forall q \geq 1$, with $||v||_q \leq C$, $\forall q \geq 1$ and Ω is bounded then $v \in L^\infty(\Omega)$.

Problem # 2. (5 marks) Let Ω be a bounded and smooth domain of $\mathbb{R}^N$ and $f \in L^2(\Omega)$. Consider the problem

$$(P_1) \qquad \begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = 1, & \text{on } \partial\Omega \end{cases}$$

i) Find the problem satisfied by $v = u - 1$

ii) Show that (P_1) has a unique weak solution $u \in H^2(\Omega)$ and give a weak formulation

Problem # 3. (15 marks)

Let Ω be a bounded and smooth domain of $\mathbb{R}^N$ and $f, g \in L^2(\Omega)$. Given the problem

$$(*) \qquad \begin{cases} -\Delta u - v + \int_\Omega u = f & \text{in } \Omega \\ -\Delta v + v + 2u = g & \text{in } \Omega \\ u = \frac{\partial v}{\partial \eta} = 0 & \text{on } \partial\Omega \end{cases}$$

i) Give a weak formulation to $(*)$

ii) Show $(*)$ has a unique weak solution $(u, v) \in (H_0^1(\Omega) \cap H^2(\Omega)) \times H^2(\Omega)$

iii) If $f = g$, show that $u - v \in H^4(\Omega)$

Problem 4 (10 marks)
Let Ω be a bounded and smooth domain of $\mathbb{R}^N$ and $f \in L^2(\Omega)$. Given the problem

$$(**) \qquad \begin{cases} \Delta^2 u = f & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases}$$

Show that $(**)$ has a unique weak solution $u \in H_0^1(\Omega) \cap H^4(\Omega)$
Hint: You may take of $v = -\Delta u$

Problem 5 (5 marks)

Let Ω be a bounded and smooth domain of $\mathbb{R}^N$ and $f \in L^2(\Omega)$. Use the maximum principle to show that

$$(***) \qquad \begin{cases} -\Delta u + u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

has at most one weak solution $u \in H^1(\Omega) \cap C(\overline{\Omega})$, where g is $C(\partial\Omega)$.