King Fahd University of Petroleum & Minerals
Department of Math. & Stat.

Math 690 - Midterm Exam 1 (131) Time: 3 H 00 min

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Name: ID #
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Problem 1	/10
Problem 2	/15
Problem 3	/10
Problem 4	/10
Problem 5	/15
Total	/60

Problem # 1. (5 marks) Let (u_n) be a sequence in $L^p(\Omega)$, $1 \leq p < +\infty$ such that $u_n \to u$ in $L^p(\Omega)$ and (v_n) be a bounded sequence in $L^\infty(\Omega)$ such that $v_n \to v$ a.e.

i) Show that $\int u_n v_n \to \int uv$ in $L^p(\Omega)$

ii) Give an example to show that the boundedness condition on (v_n) is essential.

Problem # 2. (5 marks) Suppose that (u_n) is a bounded sequence in $W^{1,p}(I)$ such that $u_n \to u$ in $L^p(\Omega)$, $1 < p < +\infty$.

i) Show that $u \in W^{1,p}(I)$.

ii) How about $p = 1$? **Hint**: consider

$$u_n(x) = \begin{cases} x^n, & 0 < x < 1 \\ 1, & 1 \leq x < 2 \end{cases}$$

iii) If I is bounded show that (i) holds for $p = +\infty$

iv) (**Do it home**) How about I is unbounded and $p = +\infty$?

Problem # 3. (5 marks) On R, let

$$\rho(x) = \begin{cases} (x^2-1)^2, & -1 < x < 1 \\ 0, & \leq |x| \geq 1 \end{cases}$$

a. Verify that $\rho \in C_0^1(R)$
b. Construct a sequence (ρ_n) satisfying

$$\rho_n \in C_0^1(R),\ \operatorname{supp}\rho_n \subset [-\frac{1}{n}, \frac{1}{n}] \text{ and } \int_{-\infty}^{\infty} \rho_n(x)dx = 1$$

c. Let $u \in W^{1,p}(R)$, show that $u_n = \rho_n * u \to u$ in $L^\infty(R)$.

Problem # 4. Define the linear functional F on $H_0^1((0,2))$by

$$F(u) = \int_0^1 fu, \qquad f \in L^2((0,2))$$

a. Show that F is well defined and bounded
b. Show that there exists $g \in L^2((0,2))$ such that

$$F(u) = \int_0^2 gu', \ \forall u \in H_0^1((0,2)).$$

and find g when $f(x) = x^2$

Problem # 5. (10 marks) Let $I = (0,1)$ and

$$H_0^2(I) = \left\{ v \in H^2(I) \;/\; v(0) = v(1) = v'(0) = v'(1) = 0 \right\}$$

a. Show that $||u|| = ||u''||_2$ defines an equivalent norm on $H_0^2(I)$
b. Give a weak formulation of the problem

$$\text{(P)} \qquad \begin{cases} -u^{(4)} + \int_0^1 u(t)dt = f & \text{in } I \\ u(0) = u'(0) = u(1) = u'(1) = 0 \end{cases}$$

when $f \in L^2(I)$ and show that (P) has a unique weak solution $u \in H_0^2(I)$.
c. Show that $u \in H^4(I) \cap H_0^2(I)$
d. Find u, if $f(x) = 24$.