Final Exam- Math 572- Term 131

Name:

(1) Consider the Dirichlet boundary value problem

$$-T''(t) = f(t), \quad t \in (0,1),$$
$$T(0) = a, \quad T(1) = b,$$

where a and b are constants, and f a polynomial.

(a) Write a weak formulation, and then apply the Galerkin method to formulate a quadratic, finite element method for the above boundary value problem.

(b) Derive the corresponding finite dimensional linear system $AU = b$. Calculate all integrations involved in the matrix A using the Simpson's Formula.

(c) Derive the Galerkin method error estimate $\|T - T_h\|_V$.

(d) Find the error when f is constant.

(2) Let $\Omega \subset R^n$ be a regular, bounded domain and $a(x) \geq a_0 > 0$, b, g be smooth functions. Consider the Dirichlet problem, assuming it has a solution: Find $u \in C^2(\overline{\Omega})$ such that

$$-\nabla \cdot (a\nabla u) + b \cdot \nabla u = 0, \qquad in \quad \Omega$$
$$u = g, \qquad on \quad \partial\Omega.$$

(a) Derive the max/min principle.

(b) Derive the stability estimate with respect to the maximum norm of $C(\Omega)$.

(c) Write a weak formulation for the above problem and show that it has a unique solution, assuming that $\nabla \cdot b \leq 0$.

(d) In the above problem, take $\Omega = (0,1) \times (0,1)$, $a = 1$, $b = (0,0)^t$, $g = x+y$ to write a finite difference scheme. Use the discrete stability estimate to show that the associated linear system has a unique solution.

(3) Consider the mixed initial-boundary value problem: Find $u(x,t)$ a solution of

$$\begin{aligned} \partial_t u - \partial_{xx} u = 0, &\quad in \quad \Omega = (0,1), \quad t > 0, \\ u(0,t) = a_0, \; u(1,t) = a_1, &\quad t > 0, \\ u(x,0) = v(x), &\quad in \quad \Omega, \end{aligned}$$

where a_0 and a_1 are constants.

(a) Derive the forward Euler scheme for the above problem and write the corresponding linear system.

(b) Let $\lambda = \frac{k}{h^2}$, where h and k are the step sizes in x and t, respectively. Show that the forward Euler scheme is not stable when $\lambda > 1/2$.

(c) Introduce a numerical scheme and prove that it is stable for arbitrary choice of h and k, and comment about the order of approximation.

(4) Let u_i, $i = 1, 2$, be the weak solutions of the problems

$$\begin{aligned} -\nabla \cdot (a_i \nabla u_i) &= f \quad in \quad \Omega = (a,b)^n, \\ u_i &= 0 \quad on \quad \partial\Omega, \end{aligned}$$

where $f \in L^2(\Omega)$, and the coefficients a_i, $a_i(x) \geq a_0 > 0$ for all $x \in \Omega$, are smooth functions.

(a) Show that

$$\int_\Omega a_i \nabla u_i \cdot \nabla(u_1 - u_2) = \int_\Omega (a_2 - a_1) \nabla u_1 \cdot \nabla u_2.$$

(b) Prove the stability estimate with respect to the coefficients

$$|u_1 - u_2|_{H^1(\Omega)} \leq \frac{C}{a_0^2} \|a_1 - a_2\|_{C(\Omega)} \|f\|_{L^2(\Omega)}.$$

Bonus Questions

(5) Let $\Omega \subset R^n$.

(a) Define the space of distributions $D'(\Omega)$.

(b) The Dirac's delta functional δ is defined on $D(\Omega)$ by

$$\delta(\varphi) = \varphi(\mathbf{0}).$$

Show that $\delta \in D'(\Omega)$. Also, find $D^{(1,2,\dots,n)}\delta$ in the sense of distribution.

(6) Let $\mathbf{F}(x, y) := -D\nabla c(x, y) + c(x, y)\mathbf{u}(x, y)$ denote the mass flux of some fluid in a region Ω, where $\mathbf{u}(x, y)$ and $c(x, y)$ are the velocity and concentration of the fluid at (x, y), respectively.

Explain the physical meaning of:

(a) the term $-D\nabla c(x, y)$

(b) the term $c(x, y)\mathbf{u}(x, y)$

(c) the equation $div\,\mathbf{F}(x, y) = 0$