Exam II- Math 572- Term 131-gOoD lUcK

Name:

Q 1: Let $f(x) := \begin{cases} \sin(x): & x < 0, \\ \cos(x): & x > 0. \end{cases}$

a) Prove that f is a distribution.

b) Find f' in the sense of distribution.

Q 2: Consider the Dirichlet/Neumann problem:

$$\begin{aligned} -u''(x) + (1+x)u(x) &= 1, \quad in \;\; D'(0,1), \\ u(0) = 0, \quad u'(1) &= 0. \end{aligned} \tag{1}$$

a) Use the Lax-Milgram Theorem to show there exists a unique solution $u \in H^1(0,1)$.

b) Show that $u \in H^3(0,1)$.

Q 3: Define the bilinear form

$$a(u, w) = \int_0^1 [u'(x)w'(x) + xu'(x)w(x) + u(x)w(x))]\, dx$$

on $H_0^1(0, 1)$. Show that $a(\cdot, \cdot)$ is coercive.

Q 4: Consider the following BVP:

$$\begin{aligned} -u''(x) &= 1, \quad in \ \ \Omega = (0,1), \\ u(0) &= \alpha, \quad u'(1) = \beta, \end{aligned} \tag{2}$$

where α and β are some given, real constants.

a) Formulate a finite element problem corresponding to the above problem using the hat basis functions.

b) Write the corresponding linear system ($AU = b$), calculating all integrals involved.

Q 5: Let

$$Au := -\Delta u, \quad u \in H_0^1(\Omega), \quad \Omega \subset R^n.$$

Show that if λ_1 and λ_2 are different eigenvalues of A, then the corresponding eigenfunctions must be orthogonal.

Q 6: Let $f \in L^2(0,1)$. Write a variational formulation for the mixed problem

$$
\begin{aligned}
q + u_x &= 0, \qquad && in \ \ (0,1),\\
q_x &= f, \qquad && in \ \ (0,1),\\
u(0) = u(1) &= 0.
\end{aligned}
$$

Q 7: Consider the following problem:

$$U'(t) + AU(t) = F(t), \quad t > 0,$$
$$U(0) = V \in R^N,$$

where A is an $N \times N$ matrix and F some smooth function. Show that the stability estimate:

$$|U(t)| \leq e^{|A|T} \left(|V| + \int_0^T |F(s)| ds \right), \quad for \;\; 0 \leq t \leq T,$$

implies the uniqueness of the solution U.