KFUPM, DEPARTMENT OF MATHEMATICS AND STATISTICS

MATH 555: FINAL EXAM, SEMESTER (131), DECEMBER 28, 2013

Time: 07:00 to 10:00 pm

Exercise 1. Let A be a ring, $\mathfrak{a}$ an ideal contained in the Jacobson radical of A. Let M be an A-module, N be a finitely generated A-module, and let $u : M \longrightarrow N$ be momorphism of A modules. Show that if the induced homomorphism $M/\mathfrak{a}M \longrightarrow N/\mathfrak{a}N$ is onto, then u is onto.

Exercise 2. Let p be a prime number and $R = (\mathbb{Z}/p^n\mathbb{Z})[X]$. Show that R is a 1-dimensional Noetherian ring.

Exercise 3. Let R be an integral domain. Show that R is integrally closed if and only if $(\mathfrak{a} : \mathfrak{a}) = R$, for each finitely generated fractional ideal $\mathfrak{a}$ of R.

Exercise 4. Let R be a ring. Show that $\mathrm{Spec}(R)$ is connected if and only if the only idempotent elements of R are 0 and 1.

Exercise 5. Let R be a ring and I be an ideal of R.

(a) Show that if I is primary, than $\mathrm{Spec}(R/I)$ is connected.
(b) Let $\mathbf{K}$ be a field, $R = \mathbf{K}[X, Y]$, and $I = (XY)$. Show that $\mathrm{Spec}(R/I)$ is connected and I is not primary.

Exercise 6. Let $\mathbf{K}$ be a field and $\mathfrak{a} = (XYZ, X^2Z)$ in the ring $\mathbf{K}[X, Y, Z]$. Find a minimal primary decomposition of $\mathfrak{a}$.

Exercise 7. Let $R = \mathbb{Z}[X]$ and the ideal $\mathfrak{m} = (2, X)$.

(a) Show that $\mathfrak{m}$ is a maximal ideal of R.
(b) Show that $\mathfrak{q} = (4, X)$ is $\mathfrak{m}$-primary, but is not a power of $\mathfrak{m}$.

Exercise 8. Let R be a ring and I be an ideal of R. We equip $\mathrm{Spec}(R)$ with the Zariski topology.

(a) Show that the closed set $V(I)$ is irreducible if and only if $\sqrt{I}$ is prime.
(b) Deduce from (a) that $\mathrm{Spec}(R)$ is irreducible if and only if the nilradical $\eta(R)$ of R is a prime ideal.

Exercise 9. Let X be a topological space.

(a) Show that X is Noetherian if and only if every open set of X is compact.

(b) Show that if R is a ring and $X = \mathrm{Spec}(R)$ (endowed with the Zariski topology), then the following statements are equivalent:

(i) X is a Noetherian space;

(ii) every radical ideal of R is the radical of a finitely generated ideal.

(c) Show that, in a Noetherian space, the set of all irreducible components of X is finite.

(d) Deduce from (3) that, in a Noetherian ring R, the set of all minimal prime ideals of R is finite.

Exercise 10. Let $\mathbf{K}$ be a a field.

(a) Show that the ideal $I = (X^2, Y)$ of $R = \mathbf{K}[X, Y]$ is primary which is not a power of a prime ideal.

(b) Let $R = \mathbf{K}[X, Y, Z]/(XY - Z^2)$ and $\overline{X}, \overline{Y}, \overline{Z}$ be, respectively, the images of X, Y, Z in R (by the canonical morphism). Show that $\mathfrak{p} = (\overline{X}, \overline{Y})$ is a prime ideal of R and $\mathfrak{p}^2$ is not primary.

Exercise 11. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of the ring A.

(a) Show that if $\mathfrak{a}$ and $\mathfrak{b}$ are comaximal (i.e., $\mathfrak{a} + \mathfrak{b} = A$), then $(\mathfrak{b} : \mathfrak{a}) = \mathfrak{b}$.

(b) Show that

$$Hom_A(A/\mathfrak{a}, A/\mathfrak{b}) = (\mathfrak{b} : \mathfrak{a})/b.$$

Show that $Hom_A(A/x.A, A) = Ann(x)$ for any $x \in A$.

(c) Show that if $\mathfrak{a}$ and $\mathfrak{b}$ are comaximal, then $Hom_A(A/\mathfrak{a}, A/\mathfrak{b}) = 0$.

(d) What is $Hom_{\mathbb{Z}}(\mathbb{Z}/55\mathbb{Z}, \mathbb{Z}/121\mathbb{Z})$? What about $Hom_{\mathbb{Z}}(\mathbb{Z}/55\mathbb{Z}, \mathbb{Z}/565\mathbb{Z})$?

Exercise 12. Let R be a ring with finite Krull dimension.

(a) Show that if P is a prime ideal of $R[X]$, $\mathfrak{p} = P \cap R$ such that $\mathfrak{p}[X] \subset P$, then $\mathrm{ht}(P) = \mathrm{ht}(\mathfrak{p}[X]) + 1$ (use induction on $\mathrm{ht}(P)$.

(b) Show that if $\mathrm{ht}(\mathfrak{p}[X]) = \mathrm{ht}(\mathfrak{p})$, for each prime ideal $\mathfrak{p}$ of R, then $dim(R[X]) = dim(R) + 1$.