KFUPM, DEPARTMENT OF MATHEMATICS AND STATISTICS

MATH 555: EXAM I, SEMESTER (131), NOVEMBER 24, 2013

Time: 08:00 to 10:00 am

Exercise 1. Show that there is no ring homomorphism from $\mathbb{Z}[\sqrt{2}]$ into $\mathbb{Z}[\sqrt{3}]$.

Exercise 2. Show that every subring of $\mathbb{Q}$ is of the form $S^{-1}\mathbb{Z}$, for some multiplicatively closed set S of $\mathbb{Z}$.

Exercise 3. Let A be a commutative ring, and let I, J be two (proper) ideals such that every prime ideal of A contains either I or J but no prime ideal contains both I and J. Prove that $A \cong A_1 \times A_2$ for some (nontrivial) rings A_1 and A_2.

Exercise 4. Let K be a field and $A = K[X, Y]/(X^2, XY, Y^2)$.

(a) Find the invertible elements of A.
(b) Find all the principal ideals of A.
(c) Find all the ideals of A.

Exercise 5. Let I be a nilpotent ideal in a ring R, M and N be R-modules, and $f : M \longrightarrow N$ be an R-homomorphism. Show that if the induced map $\bar{f} : M/IM \longrightarrow N/IN$ is surjective, then f is surjective.

Exercise 6. Let I, J be two ideals of a ring R. Show that

$$(R/I) \otimes_R (R/J) \cong R/(I + J).$$

Exercise 7. Let K be a field. Show that $K[X] \otimes_K K[Y] \cong K[X, Y]$.

Exercise 8. Let $f : A \longrightarrow B$ be a morphism of rings and J be an ideal of B and $\mathfrak{q}$ be a prime ideal of B. Show that if J is $\mathfrak{q}$-primary, then $f^{-1}(J)$ is $f^{-1}(\mathfrak{q})$-primary.

Exercise 9. Let A be a ring and let $A[X]$ be the ring of polynomials in an indeterminate X, with coefficients in A. Let $f = a_0 + a_1X + ... + a_nX^n \in A[X]$. Prove that:

(a) f is a unit in $A[X]$ if and only if a_0 is a unit in A and $a_1, \dots, a_n$ are nilpotent.
(b) f is nilpotent if and only if $a_0, a_1, \dots, a_n$ are nilpotent.
(c) f is a zero-divisor if and only if there exists $a \neq 0$ in A such that $af = 0$.