King Fahd University of Petroleum & Minerals
Department of Mathematics and Statistics

Math 550 – Linear Algebra (Term 131)

Final Exam

Notes:

- Math Students: Solve **6** problems (out of 7).
- Non-Math Students: Solve **5** problems (out of 7).
- Duration = **3 hours**.
- Each problem is worth **20 points**.

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(1) Let T be a linear operator on $\mathbb{R}^3$ which is represented in the standard ordered basis by the matrix $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

(a) Prove that T has no cyclic vector.

(b) What is the T-cyclic subspace generated by the vector $\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$.

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(2) Let n be an integer ≥ 2 and let N be an $n \times n$ matrix over the field F such that $N^n = 0$ but $N^{n-1} \neq 0$. Prove that N has no square root (i.e., there is no $n \times n$ matrix A such that $A^2 = N$).

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(3) Let A be an $n \times n$ matrix with real entries such that $A^2 + I = 0$.

(a) Prove n is even.

Let n = 6.

(b) Give the <u>rational</u> form of A.

(c) Describe explicitly the <u>cyclic</u> decomposition.

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(4) Give all possible <u>Jordan</u> forms for linear operator T with minimal polynomial $x^2(x-1)^2$ and characteristic polynomial $x^3(x-1)^4$.

(5) Let V be the vector space of the polynomials over $\mathbb{R}$ of degree ≤ 3, with the inner product $(f|g) = \int_0^1 f(x)\, g(x) dx$.

Let t be a real number, find explicitly $g_t \in$ V such that $(f|g_t) = f(t)$ for all f in V.

(Hint: Use Theorem 6 of Section 8.3)

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(6) Let V be a finite-dimensional inner product space. For each α, β in V, let $T_{\alpha,\beta}$ be the linear operator on V defined by $T_{\alpha,\beta}(\gamma) = (\gamma|\beta)\alpha$. Show that

(a) $T^*_{\alpha,\beta} = T_{\beta,\alpha}$

(b) Trace($T_{\alpha,\beta}$) = $(\alpha|\beta)$

(c) $T_{\alpha,\beta} T_{\gamma,\delta} = T_{\alpha,(\beta|\gamma)\delta}$

(d) Under what conditions is $T_{\alpha,\beta}$ self-adjoint?

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(7) Let f and g be bilinear forms on a finite dimensional vector space V. Suppose that g is non-singular.

(a) Show that there exist unique linear operators T_1 and T_2 on V such that:

$$f(\alpha, \beta) = g(T_1\alpha, \beta) = g(\alpha, T_2\beta) \quad \text{for all } \alpha, \beta.$$

(Hint: For a bilinear form $\boldsymbol{\varphi}$ use: $\boldsymbol{\varphi(\alpha,\beta) = X^t A Y}$ where $\boldsymbol{[\varphi] = A, [\alpha] = X, [\beta] = Y}$)

(b) Explain why (a) is not true if g is singular.

-------------------------------------- Good Luck --