King Fahd University of Petroleum & Minerals
Department of Mathematics and Statistics

Math 550 – Linear Algebra (Term 131)

Midterm Exam

Notes:

- Math Students: Solve **5** problems. Duration = **3 hours**.
- Non-Math Students: Solve **3** problems. Duration = **2 hours**.
- Each student should solve either **(3-1)** or **(3-2)**.
- Each problem is worth **20 points**.

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(1) Let T be the linear operator on $\mathbf{R}^2$ defined by:

$$T(x, y) = (-y, x)$$

- **(a)** What is the matrix of T in the standard ordered basis for $\mathbf{R}^2$?
- **(b)** What is the matrix of T in the ordered basis B $= \{\alpha, \beta\}$, where $\alpha = (1, 2)$ and $\beta = (1, -1)$.
- **(c)** Prove that for every real number c the operator $(T - cI)$ is invertible.
- **(d)** Prove that if B is any ordered basis for $\mathbf{R}^2$ and $[T]_B = A$, then $A_{12}A_{21} \neq 0$.

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(2) Let

$$A = \begin{bmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{bmatrix}$$

- **(a)** Is A similar over the field $\mathbb{R}$ to a diagonal matrix?
- **(b)** Is A similar over the field $\mathbb{C}$ to a diagonal matrix?

(3-1) Let V be the vector space of continuous real-valued functions on the interval [-1, 1] of the real line. Let W_e be the subspace of even functions, $f(-x) = f(x)$, and let W_o be the subspace of odd functions, $f(-x) = -f(x)$. Let T be the indefinite integral operator

$$Tf(x) = \int_0^x f(t)dt$$

- **(a)** Show that $V = W_e \oplus W_o$.
- **(b)** Is W_e invariant under T?
- **(c)** Is W_o invariant under T?

(3-2) Let V be the vector space of $n \times n$ matrices over the field $\boldsymbol{F}$.

- **(a)** If B is a fixed $n \times n$ matrix, define a function f_B on V by $f_B(A) = trace(BA)$. show that f_B is a linear functional on V.
- **(b)** Show that every linear functional on V is of the above form, i.e., is f_B for some B.
- **(c)** Show that $B \rightarrow f_B$ is an isomorphism of V onto V*.

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(4) Let V be the space of $n \times n$ matrices over **F**, A a fixed $n \times n$ matrix over **F**, and T the linear operators on V defined by:

$$T(B) = AB$$

- **(a)** Is it true that A and T have the same characteristic values?
- **(b)** Show that the minimal polynomial for T is the minimal polynomial for A.
- **(c)** True or False? If A is diagonalizable (over **F**), then T is diagonalizable.

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(5) Let V be the space of $n \times n$ matrices over **F**, A a fixed $n \times n$ matrix over **F**, and T the linear operator on V defined by:

$$T_A(B) = AB - BA.$$

- **(a)** True or False? If A is diagonalizable (over **F**), then T_A is diagonalizable.
- **(b)** Consider the family of linear operators T_A obtained by letting A vary over all diagonal matrices. Prove that the operators in that family are *simultaneously* diagonalizable.
- **(c)** Prove that if A is a nilpotent matrix, then T_A is a nilpotent operator.

------------------------------------- Good Luck ---