King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
MATH311 - Advanced Calculus I
Final Exam – Term 131 (2013–2014)

Exercise 1 (5 points)
Let f be continuous on the interval $[0,1]$ to $\mathbb{R}$ and such that $f(0) = f(1)$. Prove that there exists a point $c \in [0, \frac{1}{2}]$ such that $f(c) = f(c + \frac{1}{2})$.

Exercise 2 (8 points)

1. Let $f : A \to \mathbb{R}$ be a uniformly continuous on a subset A of $\mathbb{R}$.
   (a) Prove that if (x_n) is a Cauchy sequence in A, then $(f(x_n))$ is a Cauchy sequence in $\mathbb{R}$.
   (b) Prove that if A is bounded, then f is bounded on A.
2. Show that the function $f(x) = \dfrac{1}{x}$ is uniformly continuous on $[1,\infty)$ but that it is not uniformly continuous on $(0,1)$.

Exercise 3 (8 points)

1. Prove that $1 + \dfrac{x}{3} - \dfrac{x^2}{9} < (1+x)^{1/3} < 1 + \dfrac{x}{3}$ if $x > 0$.
2. Prove that $\dfrac{2}{\pi}x \leq \sin x \leq x$ if $x \in [0, \pi/2]$.

Exercise 4 (8 points) Let $p_n(x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} + \ldots + (-1)^{n-1}\dfrac{x^n}{n}$. Prove that

1. $|\ln(1+x) - p_n(x)| \leq \dfrac{1}{(x+1)^{n+1}} . \dfrac{|x|^{n+1}}{n+1}$ if $x \in (-1,0]$,
2. $|\ln(1+x) - p_n(x)| \leq \dfrac{x^{n+1}}{n+1}$ if $x \in [0,1]$.

Exercise 5 (10 points)

1. Let F and G be continuous on $[a,b]$ and differentiable on (a,b). Show that there exists $c \in (a,b)$ such that
$$(F(b) - F(a))\, G'(c) = (G(b) - G(a))\, F'(c).$$
   Hit : Consider $H(x) = [F(b) - F(a)]\,[G(x) - G(a)] - [G(b) - G(a)]\,[F(x) - F(a)]$.
2. Let $n \in \mathbb{N}$, $f, g \in C^n([a,b])$ such that $f^{(n+1)}$ and $g^{(n+1)}$ exist and are continuous on the open interval (a,b).
   (a) Let $F(x) := \displaystyle\sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}(b-x)^k$. Prove that F is continuous on $[a,b]$ and differentiable on (a,b), $F(b) = f(b)$, and
$$F'(x) = \frac{f^{(n+1)}(x)}{n!}(b-x)^n.$$

(b) Prove that there exits $c \in (a,b)$ such that

$$\left[f(b) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(b-a)^k\right] g^{(n+1)}(c) = \left[g(b) - \sum_{k=0}^{n} \frac{g^{(k)}(a)}{k!}(b-a)^k\right] f^{(n+1)}(c).$$

(c) Application: Let $I \subset \mathbb{R}$ be an interval and let $a \in I$. Suppose that the derivatives $f^{(n+1)}$, $g^{(n+1)}$ exist and are continuous on I. If $f^{(k)}(a) = 0$ and $g^{(k)}(a) = 0$ for $k = 0, 1, \ldots n$, but $g^{(n+1)}(a) \neq 0$ Show that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f^{(n+1)}(a)}{g^{(n+1)}(a)}.$$

Exercise 6 (`8 points`)
Fix a positive number α and consider the series

$$\sum_{k \geq 2} \frac{1}{k^\alpha \ln k}$$

For what values of α does this series converge?

Exercise 7 (`8 points`)

1. If the sequence of partial sums s_n of $\sum_{n=1}^{\infty} a_n$ satisfies $\lim_{n \to \infty} \frac{s_n}{\sqrt{n}} = 0$, show that the series $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges to $\sum_{n=1}^{\infty} \frac{s_n}{n(n+1)}$.

2. Find

$$\sum_{n=1}^{\infty} \frac{\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n+1}\right)}{n(n+1)}.$$

Exercise 8 (`7 points`)
Suppose that none of the numbers a, b, c is a negative integer or zero. Prove that the hypergeometric series

$$\frac{ab}{1!c} + \frac{a(a+1)b(b+1)}{2!c(c+1)} + \frac{a(a+1)(a+2)b(b+1)(b+2)}{3!c(c+1)(c+2)} + \ldots$$

is absolutely convergent for $c > a + b$ and divergent for $c < a + b$.

Exercise 9 (`8 points`)
Let $f_n(x) = x^n - x^{2n}$ for $0 \leq x \leq 1$ and $n \in \mathbb{N}$. Does the sequence (f_n) converge pointwise on $[0, 1]$? Is the convergence uniform?