King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
MATH311 - Advanced Calculus I
Exam I – Term 131 (2013–2014)

Exercise 1 (`6 points`)
State without proof

1. the Archimedean property of $\mathbb{R}$,
2. the Bolzano-Weierstrass Theorem.

Define

1. the completeness property of $\mathbb{R}$,
2. a Cauchy sequence.

Exercise 2 (`5 points`)
Prove that between any two real numbers there is a rational number.

Exercise 3 (`5 points`)
Let S be a nonempty subset of $\mathbb{R}$ that is bounded above, and let a be any real number in $\mathbb{R}$. Define the set $a + S = \{a + s \, : \, s \in S\}$. Show that

$$\sup(a + S) = a + \sup S.$$

Exercise 4 (`6 points`)
Let $\{I_n = [a_n, b_n] \, : \, n \in \mathbb{N}\}$ be a sequence of closed bounded intervals in $\mathbb{R}$ that is nested; i.e., $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$.
If $\alpha = \sup\{a_n \, : \, n \in \mathbb{N}\}$ and $\beta = \inf\{b_n \, : \, n \in \mathbb{N}\}$, show that

$$\bigcap_{n \geq 1} [a_n, b_n] = [\alpha, \beta]$$

Exercise 5 (`5 points`)
Prove that if $\lim(x_n) = x$ and if $x > 0$, then there exists a natural number K such that $x_n > 0$ for all $n \geq K$.

Exercise 6 (`7 points`)
Let $x_n = \dfrac{1}{n+1} + \dfrac{1}{n+2} + \ldots + \dfrac{1}{2n}$ for $n \in \mathbb{N}$.

1. Compute x_1 and x_2.
2. Prove that (x_n) is monotone and bounded and hence converges.

Exercise 7 (`7 points`)

1. Provide an example of a sequence (x_n) that satisfies $\lim\limits_{n\to\infty} |x_{n+1} - x_n| = 0$ but that is not a Cauchy sequence.
2. If $0 < r < 1$ and $|x_{n+1} - x_n| < r^n$ for all $n \in \mathbb{N}$, show that (x_n) is a Cauchy sequence.

Exercise 8 (`9 points`)
Let $0 < r < 1$ and y_1, y_2 be two real numbers such $y_1 < y_2$ and

$$y_n = (1 - r)y_{n-1} + r y_{n-2} \text{ for } n > 2.$$

Show that the sequence (y_n) is convergent. What is its limit?