1. (12 points) Determine two linearly independent power series solutions of the differential equation $y'' - 2xy' - 4y = 0$ about the ordinary point $x = 0$. (Write only first four terms of each solution). Hence write the general solution of the differential equation.

2. (10 points) Determine the singular points of the differential equation $2x(x-2)^2y'' + 3xy' + (x-2)y = 0$ and classify them as regular or irregular.

3. (12 points) Solve the system

$$X'(t) = \begin{pmatrix} -1 & 0 & 0 \\ 3 & -1 & 0 \\ 4 & 2 & -1 \end{pmatrix} X(t)$$

4. (12 points) Solve the initial value problem

$$X' = \begin{pmatrix} -3 & -2 \\ 4 & 1 \end{pmatrix} X, \qquad X(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

5. (12 points) Let $X_1 = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$ and $X_2 = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$ be solutions of the system $X' = AX$, for some 2×2 matrix A.
Use variation of parameters to solve the system

$$X' = AX + \begin{pmatrix} 0 \\ \sec t \end{pmatrix}.$$

6. (12 points) Use matrix exponential method to solve the initial value problem

$$X' = \begin{bmatrix} 2 & -3 & 2 \\ 0 & 0 & 0 \\ -2 & 5 & -2 \end{bmatrix} X, \qquad X(1) = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}.$$

7. (7 points) Using the existence and uniqueness theorem, the initial value problem

$$\frac{dy}{dx} = \sqrt{y - x}, \quad y(x_0) = y_0$$

is guaranteed to have a unique solution if $(x_0, y_0) =$

a) $(0, 1)$
b) $(0, 0)$
c) $(2, 2)$
d) $(3, 0)$
e) $(-1, -2)$

8. (7 points) The solution of the differential equation

$$y \cos y \frac{dy}{dx} = x(2 \ln x + 1) - \sin y \frac{dy}{dx}$$

is

a) $y \sin y = x^2 \ln x + c$
b) $y \cos y = x^2 \ln x + c$
c) $y^2 \sin y = x \ln x + c$
d) $y^2 \cos x = x \ln y + c$
e) $y^2 \cos y = x \ln y + c$

9. (7 points) The general solution of the differential equation $\dfrac{dy}{dx}+\dfrac{y}{x}=\sin x$ is:

   a) $x(y+\cos x)=\sin x+c$
   b) $x(y+\cos x)=\cos x+c$
   c) $x(y-\cos x)=\sin x+c$
   d) $x(y-\sin x)=\cos x+c$
   e) $x(y+\cos x)=\sin^{-1}x+c$

10. (7 points) A possible integration factor that will make the ordinary differential equation

$$(y^2+e^y)\,dx+(2y+e^y)\tan x\,dy=0, \qquad 0<x<\frac{\pi}{2}$$

exact equation is

   a) $\mu(x)=\cos x$
   b) $\mu(y)=\sin y$
   c) $\mu(x)=x^2\sin x$
   d) $\mu(y)=e^y\sin y$
   e) $\mu(x)=e^x+x$

11. (7 points) The solution of $\dfrac{dy}{dx} = 1 + e^{y-x+5}$, is given by:

a) $y = x - 5 - \ln(c - x)$
b) $y = -x - 5 - \ln(c - x)$
c) $y = y^2 - x + e^5 + c$
d) $y = x + e^{y - \frac{x^2}{2} + 5} + c$
e) $y = \ln(y - x + 5 + c)$

12. (7 points) Given that $y_p = ax^2 + bx$ is a particular solution of the differential equation $2x^2y'' + 5xy' + y = x^2 - x$. Then $15a + 6b =$

a) 0
b) 1
c) 2
d) 3
e) 4

13. (7 points) Given that the general solution of the differential equation $xy'' - y' + 4x^3y = 0$ is $y(x) = Ay_1(x) + By_2(x)$ with $y_1(x) = \sin(x^2)$. Then

a) $y(0) = \dfrac{-B}{2}$

b) $y(0) = A + B$

c) $y(0) = \dfrac{A}{2}$

d) $y(0) = \dfrac{A+B}{2}$

e) $y(0) = -B$

14. (7 points) Let $f(x) = (x^2 - 1)^2e^{-2x} + 2x^3e^x \sin 2x - 6x^2e^x \cos x$. Which one of the following differential operators is an annihilator of lowest possible order of f ?

a) $(D+2)^5(D^2 - 2D + 5)^4$

b) $(D+2)^5(D^2 - 2D + 5)^4(D^2 - 2D + 5)^3$

c) $(D^2+2)^2(D^2 - 2D + 5)^3$

d) $(D^2+2)^2(D^2 - 2D + 5)^3(D^2 - 2D + 5)^2$

e) $(D+2)^4(D^2 - 2D + 5)^4$

15. (7 points) A particular solution for the ordinary differential equation

$$4y'' + 36y = \frac{1}{\sin(3x)}$$

is given by

a) $-\dfrac{1}{12}x\cos(3x) + \dfrac{1}{36}\sin(3x)\ln|\sin(3x)|$

b) $\cos(3x) + \sin(3x)$

c) $\dfrac{1}{4}x\cos(3x) - \dfrac{1}{9}\ln|\sin(3x)|$

d) $x\sin(3x) - \cos(3x)\ln|\sin(3x)|$

e) $-\dfrac{1}{12}x\cos(3x) + \dfrac{1}{36}\sin(3x)$

16. (7 points) The general solution of the differential equation

$$x^2y'' + xy' + 4y = 0$$

is

a) $y = c_1\cos\,(2\ln x) + c_2\sin\,(2\ln x)$

b) $y = c_1x\cos\,(\ln x) + c_2x\sin\,(\ln x)$

c) $y = c_1x^2\cos\,(2\ln x) + c_2x^2\sin\,(2\ln x)$

d) $y = c_1\cos\,(\sqrt{2}\ln x) + c_2\sin\,(\sqrt{2}\ln x)$

e) $y = c_1x\cos\,(\sqrt{2}\ln x) + c_2x\sin\,(\sqrt{2}\ln x)$