MATH 201 FINAL EXAM

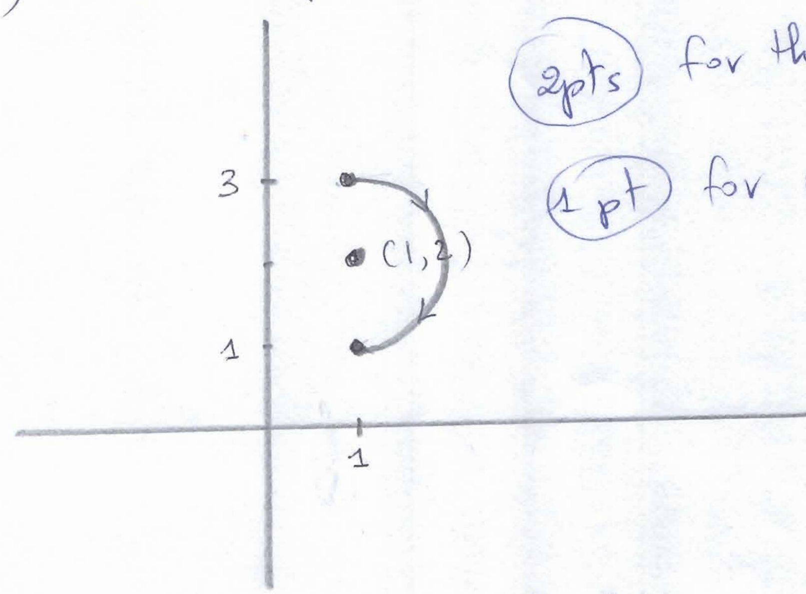
Written Part

1) A) $x = 1 + \sin t, \quad y = \cos t + 2, \quad 0 \le t \le \pi$

Eliminating the parameter, we obtain

$$(x-1)^2 + (y-2)^2 = \sin^2 t + \cos^2 t = 1$$ 2 pts

the curve is the portion of the circle centered at the point $(1,2)$ with radius 1 from the point $(1,3)$ to the point $(1,1)$ clockwise.



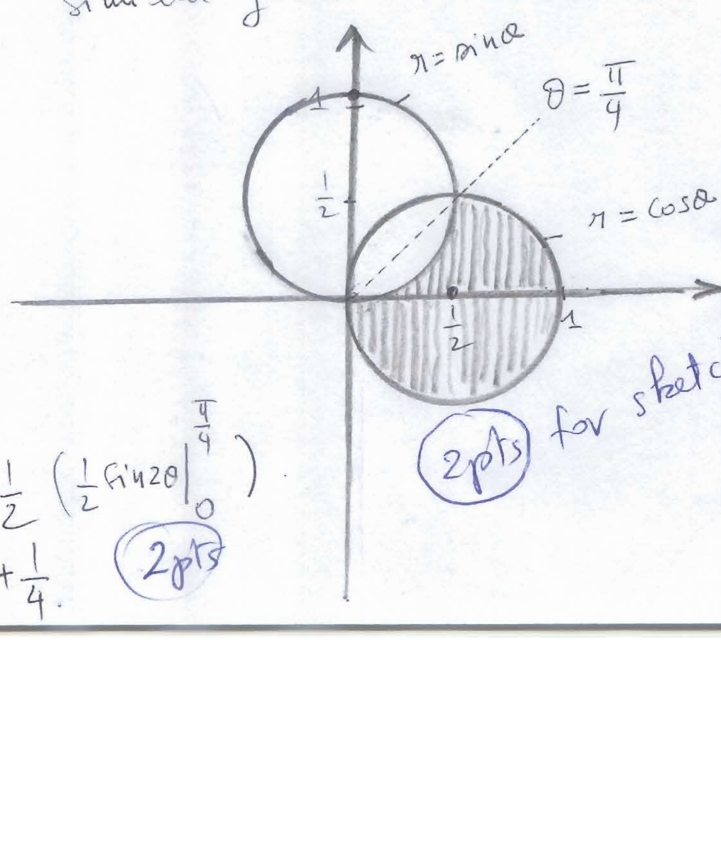
2pts for the curve

1 pt for orientation

B) $r = \cos\theta \iff r^2 = r\cos\theta \iff x^2 + y^2 = x \iff x^2 - x + y^2 = 0$

$\iff \left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}$. Similarly $r = \sin\theta \iff$

$x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$



2pts for sketch

Intersecting points

$\cos\theta = \sin\theta \iff$

$\tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$ 1 pt

$$A = \frac{\pi(1/2)^2}{2} + \frac{1}{2}\int_0^{\frac{\pi}{4}} (\cos^2\theta - \sin^2\theta)\, d\theta$$

$$= \frac{\pi}{8} + \frac{1}{2}\int_0^{\pi/4} \cos 2\theta \, d\theta = \frac{\pi}{8} + \frac{1}{2}\left(\frac{1}{2}\sin 2\theta \Big|_0^{\frac{\pi}{4}}\right)$$

$$= \frac{\pi}{8} + \frac{1}{4}.$$ 2pts

2) A) Q is given by the point of intersection of the line through Q in the direction of the normal vector to the plane ($\vec{n} = \langle 2, 1, -2 \rangle$). (2 pts)

the parametric equations of the line are

$x = 1 + 2t$, $y = 6 + t$, $z = -1 - 2t$. (2 pts)

To find Q, we have

$2(1+2t) + (6+t) - 2(-1-2t) = 19 \iff 9t = 9 \iff$

$t = 1$ (1 pt)

Substituting $t = 1$ in the parametric equations gives $Q = (3, 7, -3)$. (1 pt)

Remark. The students may do the following:

Minimize the distance $d = \sqrt{(x-1)^2 + (y-6)^2 + (z+1)^2}$

with $2x + y - 2z = 19$

→ By Lagrange Method or

→ By substituting $z = x + \frac{y}{2} - \frac{19}{2}$ in d.

In either case, they will find

$x = 3$, $y = 7$ and $z = -3$.

2B) The scalar component of $\vec{u}$ in the direction of $\vec{v}$ is given by

$$\vec{u} \cdot \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 0, 3, 4 \rangle \cdot \langle 10, 11, -2 \rangle}{\sqrt{100 + 121 + 4}}$$ (2 pts)

$$= \frac{25}{\sqrt{225}} = \frac{25}{15} = \frac{5}{3}$$ (1 pt)

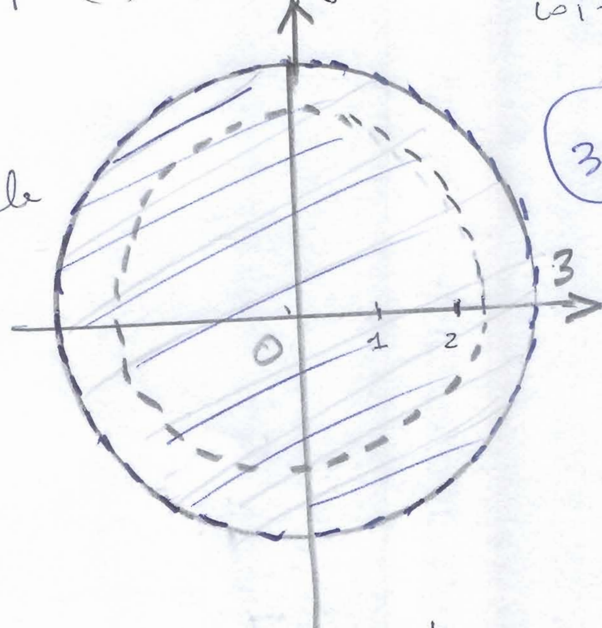
3) A) $f(x,y) = \dfrac{1}{\ln(9-x^2-y^2)}$

Domain $= \{ (x,y) \mid 9-x^2-y^2 > 0$ and $9-x^2-y^2 \neq 1 \}$ 2pts

$9-x^2-y^2 > 0 \iff x^2+y^2 < 9$: open disc centered at $(0,0)$ with radius 3

$9-x^2-y^2 = 1 \iff x^2+y^2 = 8$: Circle centered at $(0,0)$ with radius $\sqrt{8}$

Domain = Disc − circle

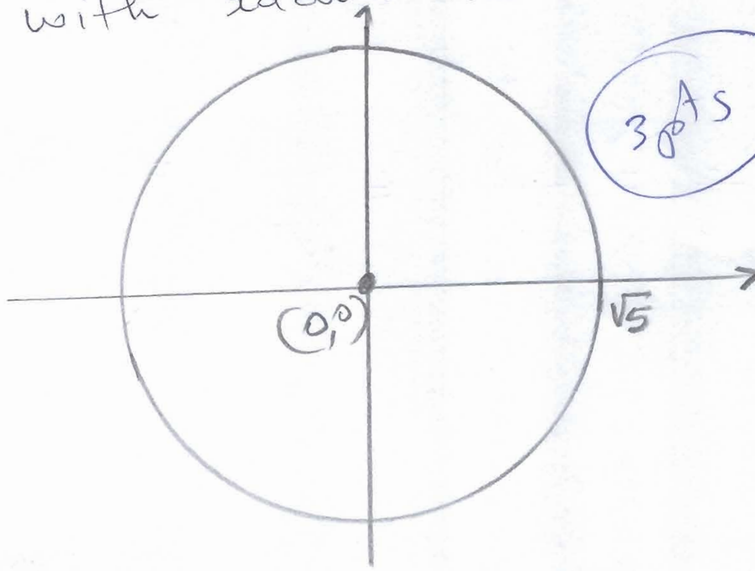


3pts

B) $f(x,y) = f(2,1) \iff \dfrac{1}{\ln(9-x^2-y^2)} = \dfrac{1}{\ln 4} \iff$

$9-x^2-y^2 = 4 \iff x^2+y^2 = 5$ 2pts

the level curve is the circle centered at $(0,0)$ with radius $\sqrt{5}$



3pts

4) $f(x,y) = x^3 + 3xy^2 - 15x + y^3 - 15y$.

* critical points :

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \iff \begin{cases} x^2 + y^2 = 5 & (1) \\ 2xy + y^2 = 5 & (2) \end{cases}$$ 1pt

$(1) - (2) \Rightarrow x^2 - 2xy = 0 \iff x(x - 2y) = 0$

$\Rightarrow x = 0$ or $x = 2y$

* if $x = 0$, $y = \pm\sqrt{5}$ and the critical points are $(0, \sqrt{5})$ 1pt and $(0, -\sqrt{5})$ 1pt

* If $x = 2y$, $4y^2 + y^2 = 5 \iff 5y^2 = 5 \Rightarrow y = \pm 1$ and the critical points are

$(2,1)$ 1pt and $(-2,-1)$ 1pt

$$D(x,y) = f_{xx} f_{yy} - f_{xy}^2 = (6x)(6x+6y) - (6y)^2$$
$$= 36\left[x(x+y) - y^2\right]$$ 1pt

$D(0, \pm\sqrt{5}) = 36(-25) < 0 \Rightarrow (0, \pm\sqrt{5})$ are saddle points 2pts

$D(2,1) = 36(6-1) > 0$, $f_{xx}(2,1) > 0 \Rightarrow f(2,1)$ is a local minimum 1pt

$D(-2,-1) = 36(6-1) > 0$, $f_{xx}(-2,-1) < 0 \Rightarrow f(-2,-1)$ is a local maximum. 1pt

5) * Critical points.

$$\begin{cases} T_x = 0 \\ T_y = 0 \end{cases} \iff \begin{cases} 2x - 1 = 0 \\ 4y = 0 \end{cases} \iff \begin{cases} x = \frac{1}{2} \\ y = 0 \end{cases}$$ 1 pt

* Lagrange multiplier : $g(x,y) = x^2 + y^2$, $c = 1$

$$\begin{cases} T_x = \lambda g_x \\ T_y = \lambda g_y \\ g = c \end{cases} \iff \begin{cases} 2x - 1 = 2\lambda x & \text{1 pt} \\ 4y = 2\lambda y & \text{1 pt} \\ x^2 + y^2 = 1 & \text{1 pt} \end{cases}$$

$4y = 2\lambda y \iff 2y(2 - \lambda) = 0 \implies y = 0$ or $\lambda = 2$

$\to$ if $y = 0$, then $x = \pm 1$ 2 pts

$\to$ if $\lambda = 2 \implies 2x - 1 = 4x \implies x = -\frac{1}{2}$ and $y = \pm \frac{\sqrt{3}}{2}$ 2 pts

* Checking Values

$T(\frac{1}{2}, 0) = -\frac{1}{4}$,

$T(1, 0) = 0$

$T(-1, 0) = 2$

$T(-\frac{1}{2}, \pm \frac{\sqrt{3}}{2}) = \frac{9}{4}$

* The maximum value is $\frac{9}{4}$ 1 pt

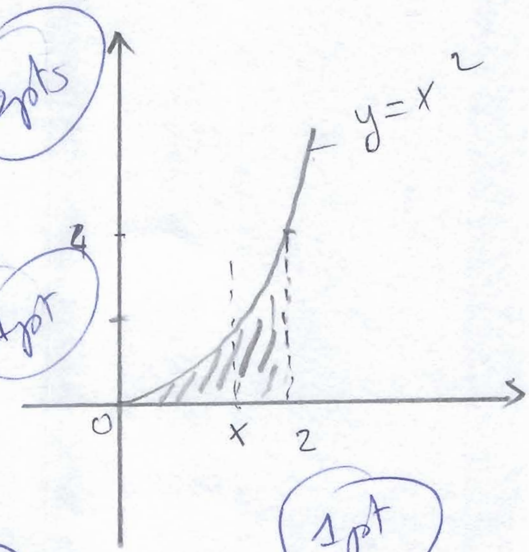
* The minimum value is $-\frac{1}{4}$ 1 pt

6) A) We need to reverse the order of integration

$$\sqrt{y} \le x \le 2 \;, \qquad 0 \le y \le 4$$

$$x = \sqrt{y} \iff y = x^2 \;, \quad x = 2 \;, \quad y = 0 \;, \quad y = 4$$

$$\int_0^4 \int_{\sqrt{y}}^{2} \frac{\sin(x^2)}{x}\, dx\, dy = \int_0^2 \int_0^{x^2} \frac{\sin(x^2)}{x}\, dy\, dx$$ 2pts



1pt

$$= \int_0^2 x \sin(x^2)\, dx$$ 1pt

$$= -\frac{1}{2} \cos(x^2) \Big|_0^2$$

$$= \frac{1 - \cos 4}{2}$$ 1pt

B) $\displaystyle\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_0^{2x+y} dz\, dx\, dy$? we have

$$\left.\begin{array}{l} 0 \le z \le 2x + y \\ -\sqrt{4-y^2} \le x \le \sqrt{4-y^2} \\ 0 \le y \le 2 \end{array}\right\} \Rightarrow$$ The integration with respect to (x, y) is over the circular region $x^2 + y^2 = 4$, $y \geq 0$

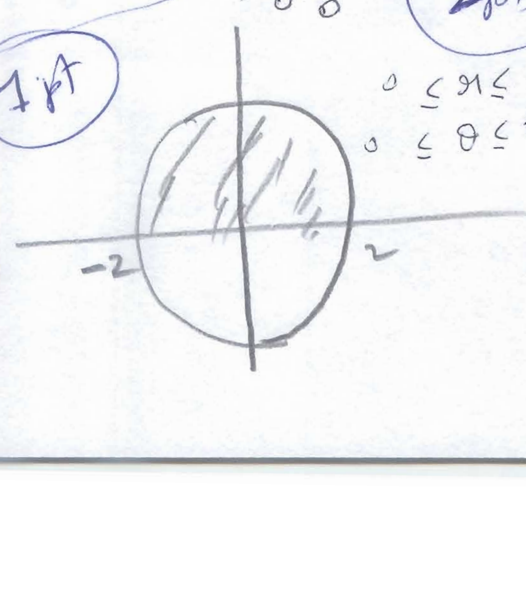
We therefore use polar coordinates for the xy-integral

$$\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_0^{2x+y} dz\, dx\, dy = \underbrace{\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (2x + y)\, dx\, dy}_{1pt} = \int_0^{\pi} \int_0^2 (2r\cos\theta + r \sin\theta)\, r\, dr\, d\theta$$ 2pts

$0 \le r \le 2$

$0 \le \theta \le \pi$

$$= \int_0^{\pi} \frac{r^3}{3} (2\cos\theta + \sin\theta) \Big|_{r=0}^{r=2} d\theta$$ 1pt



$$= \frac{8}{3} \left(2\sin\theta - \cos\theta \Big|_0^{\pi} \right) = \frac{16}{3}$$ 1pt

7) A) $\overrightarrow{PQ} = \langle 2, 1 \rangle$ is not a unit vector. We take $\vec{u} = \dfrac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \dfrac{1}{\sqrt{5}} \langle 2, 1 \rangle$

$$f_x = \frac{2x}{\sqrt{x^2+4y}}, \quad f_y = \frac{4}{\sqrt{x^2+4y}}$$ 2pts

$$D_{\vec{u}} f(-2,3) = f_x(-2,3)\left(\frac{2}{\sqrt{5}}\right) + f_y(-2,3)\left(\frac{1}{\sqrt{5}}\right)$$ 2pts

$$= \left(\frac{-4}{\sqrt{16}}\right)\left(\frac{2}{\sqrt{5}}\right) + \left(\frac{4}{\sqrt{16}}\right)\left(\frac{1}{\sqrt{5}}\right)$$

$$= -\frac{2}{\sqrt{5}} + \frac{1}{\sqrt{5}} = -\frac{1}{\sqrt{5}}$$ 1pt

B) $\nabla f(-2,3) = \langle -1, 1 \rangle$ 2pts

$|\nabla f(-2,3)| = \sqrt{2}$ $\quad \dfrac{\nabla f}{|\nabla f|}(-2,3) = \left\langle -\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}} \right\rangle$ 1pt

The directions of zero changes are

$\vec{u} = \left\langle \dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}} \right\rangle$ or $\vec{u} = \left\langle -\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}} \right\rangle$

1pt 1pt

Q	MM	V1	V2	V3	V4
1	a	b	e	d	b
2	a	c	c	d	a
3	a	b	c	d	b
4	a	e	d	a	b
5	a	d	a	b	c
6	a	b	a	b	b
7	a	a	e	b	a
8	a	b	c	b	e
9	a	c	e	c	c
10	a	d	d	b	b
11	a	a	c	a	a
12	a	d	b	e	d
13	a	e	c	c	a
14	a	e	a	c	e