King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 201 - Exam I - Term 131

Duration: 120 minutes

Key

Rasasi

Name:________________ ID Number:________________

Section Number:________________ Serial Number:________________

Class Time: ________________ Instructor's Name:________________

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write neatly and eligibly. You may lose points for messy work.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 8 pages of problems (Total of 8 Problems)

	Points	Maximum Points
page 1		13
page 2		13
page 3		12
page 4		16
page 5		12
page 6		12
page 7		8
page 8		14
Total		100

1. A) (7 points) Convert the parametric equations

$$x = 1 + 2\sin t, \qquad y = 2\cos t + 2, \qquad 0 \le t \le \frac{3\pi}{4}$$

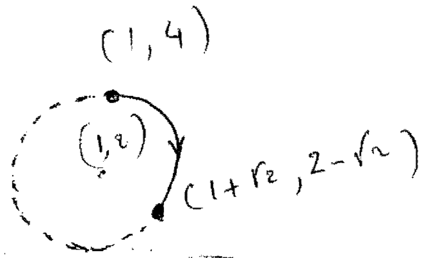
into carterian (rectangular) equation. Sketch the curve with the direction of motion.

We have $2\sin t = x - 1$ and $2\cos t = y - 2$. This implies

$(x-1)^2 + (y-2)^2 = 4(\sin^2 t + \cos^2 t) = 4 \iff$

(4 pts) $\boxed{(x-1)^2 + (y-2)^2 = 4}$. With $0 \le t \le \frac{3\pi}{4}$, the curve is the portion of the circle clockwise from the point $(1,4)$ to the point $(1+\sqrt{2}, 2-\sqrt{2})$

circle centered at $(1,2)$ with radius 2



(3 pts)

B) (6 points) Determine the concavity of the curve given by

$$x = \theta - \sin\theta, \qquad y = 1 - \cos\theta, \qquad \text{at } \theta = \pi.$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\sin\theta}{1-\cos\theta} \quad (1\text{ pt})$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left[\frac{dy}{dx}\right]}{\frac{dx}{d\theta}} = \frac{\frac{\cos\theta(1-\cos\theta) - \sin^2\theta}{(1-\cos\theta)^2}}{1-\cos\theta} \quad (2\text{ pts})$$

(2 pts)

at $\theta = \pi$ we have

$$\left.\frac{d^2y}{dx^2}\right|_{\theta=\pi} = \frac{(-1)(2) - 0}{8} = -\frac{2}{8} = -\frac{1}{4} < 0$$

The curve is concave down at $\theta = \pi$ (1 pt)

2. A) (6 points) Find an equation of the line that passes through the point $(1,1)$ and parallel to the tangent line to the curve

$$x = \ln t \qquad y = \sqrt{t+1} \qquad \text{at } t = 3.$$

slope $= \left.\frac{dy}{dx}\right|_{t=3}$. (1 pt) $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{1}{2\sqrt{t+1}}}{\frac{1}{t}} = \frac{t}{2\sqrt{t+1}}$ (2 pts)

slope at $t = 3$ is $\frac{3}{4}$ (1 pt)

using now the point - slope form, the equation of the line is given by

$y = \frac{3}{4}(x-1) + 1$ (2 pts)

$= \frac{3}{4}x + \frac{1}{4}$

B) (7 points) Let C be the portion of the curve $x = 3\cos\theta$, $y = 3\sin\theta$ from $(3,0)$ to $\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$. Find the area of the surface obtained by rotating C about the x-axis.

first observe that $(3,0) \longrightarrow t = 0$ and

$\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right) \longrightarrow t = \frac{\pi}{3}$ (2 pts)

$S = 2\pi \int_0^{\frac{\pi}{3}} 3\sin t \sqrt{(-3\sin t)^2 + (3\cos t)^2}\, dt$ (3 pts)

$= 6\pi \int_0^{\frac{\pi}{3}} \sin t \sqrt{9(\sin^2 t + \cos^2 t)}\, dt$

$= 18\pi \int_0^{\frac{\pi}{3}} \sin t\, dt = -18\pi\left(\cos t \Big|_0^{\frac{\pi}{3}} \right)$ (1 pt)

$= -18\pi\left(\frac{1}{2} - 1\right)$

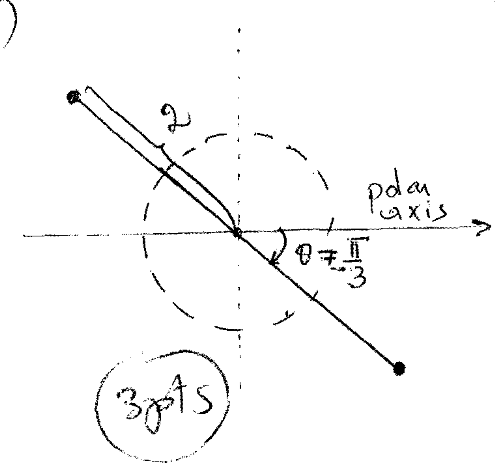
$= 9\pi$ (1 pt)

3. A)(6 points) Graph the set of points (r, θ) whose polar coordinates satisfy the given conditions:

   a) $\theta = -\dfrac{\pi}{3}$ and $-2 \le r \le 2$

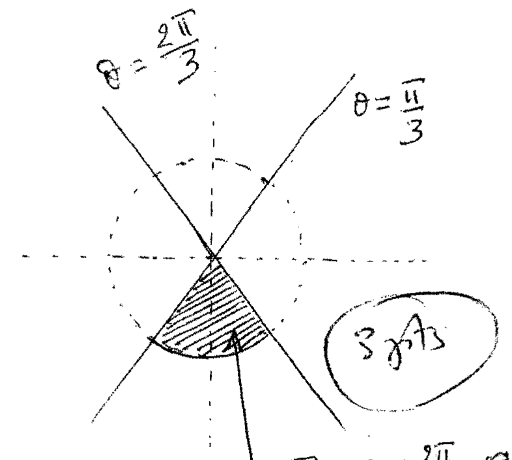
   b) $\dfrac{\pi}{3} \le \theta \le \dfrac{2\pi}{3}$ and $-1 \le r \le 0$

a)



3pts

b)



3pts

$\frac{\pi}{3} \le \theta \le \frac{2\pi}{3}$ and
$-1 \le r \le 0$

B) (6 points) Sketch the polar curves given by

   a) $r \sin\left(\theta + \dfrac{\pi}{6}\right) = 2$

   b) $r = \cot\theta \csc\theta$

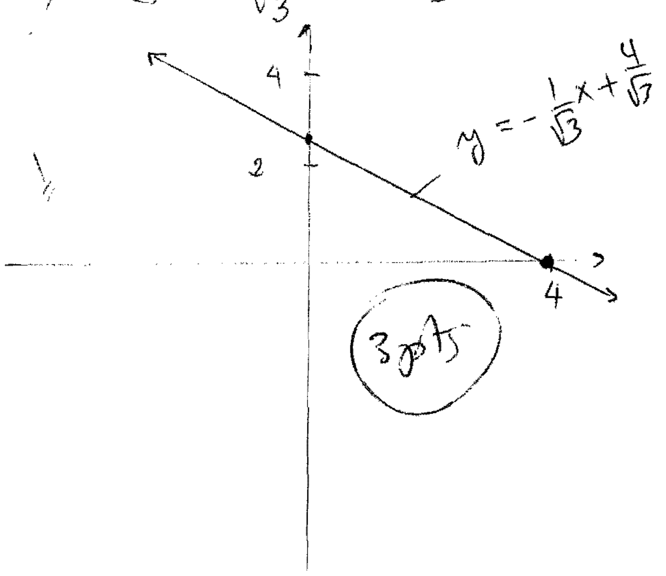
a) $r \sin\left(\theta + \frac{\pi}{6}\right) = 2 \Longleftrightarrow$

$r \sin\theta \cos\frac{\pi}{6} + r\cos\theta \sin\frac{\pi}{6} = 2$

$\Longleftrightarrow$

$\frac{\sqrt{3}}{2} y + \frac{x}{2} = 2 \Longleftrightarrow$

$y = -\frac{1}{\sqrt{3}} x + \frac{4}{\sqrt{3}}$

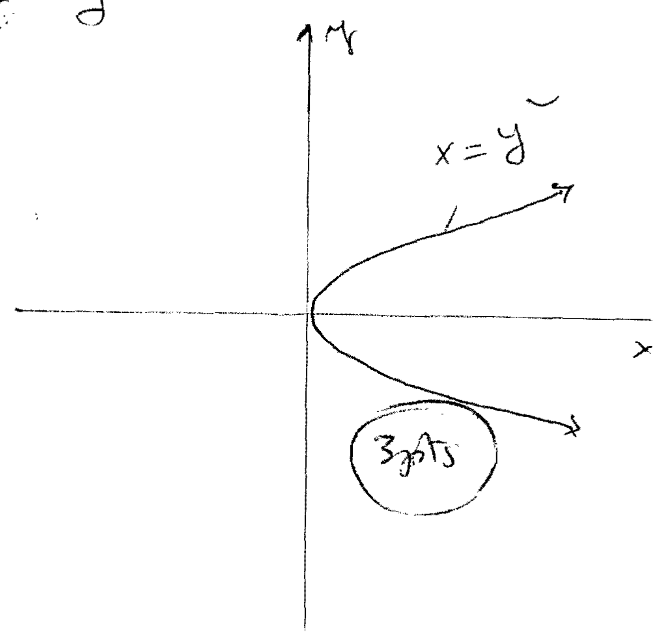


3pts

b) $r = \cot\theta \csc\theta = \dfrac{\cos\theta}{\sin^2\theta}$

$\Longleftrightarrow r^2 \sin^2\theta = r\cos\theta \Longleftrightarrow$

$y^2 = x$



3pts

4. A) (8 points) Consider the polar curve C given by $r = 1 - \cos\theta,\ 0 \le \theta \le 2\pi$. Find the slope of C at $\theta = \frac{\pi}{2}$ and the total length of C.

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{2}} = \left.\frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}\right|_{\theta=\frac{\pi}{2}} = \frac{(1)(1)+0}{(1)(0)-1} = -1$$

(1 pt) (2pts)

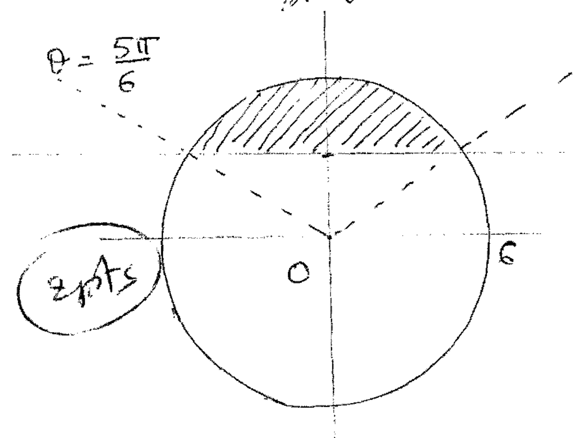
(3pts) $$s = \int_0^{2\pi} \sqrt{(1-\cos\theta)^2 + \sin^2\theta}\, d\theta = \int_0^{2\pi} \sqrt{2}\sqrt{1-\cos\theta}\, d\theta =$$

$$= \sqrt{2}\int_0^{2\pi} \sqrt{2\sin^2\left(\frac{\theta}{2}\right)}\, d\theta = 2\int_0^{2\pi} \sin\left(\frac{\theta}{2}\right) d\theta = 8$$

(1pt) (1 pt)

B) (8 points) Find the area inside the circle $r = 6$ and above the line $r = 3\csc\theta$.

$$r = 3\csc\theta \iff r = \frac{3}{\sin\theta} \iff r\sin\theta = 3 \iff y = 3$$



(2pts)

intersecting points

$6 = \frac{3}{\sin\theta} \iff \sin\theta = \frac{1}{2}$

$\Rightarrow \theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$

(2pts)

$$S = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{6^2}{2}\, d\theta - \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{9\csc^2\theta}{2}\, d\theta$$

(2pts)

$$= 18\left(\frac{5\pi}{6} - \frac{\pi}{6}\right) + \frac{9}{2}\left(\cot\theta \Big|_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\right)$$

(1 pt)

$$= 12\pi - 9\sqrt{3}$$

(1 pt)

5. A) (6 points) Give an equation of the circle of radius 2 centered at $(0, 2, 0)$ and lying in a plane parallel to the yz-plane.

$(y-2)^2 + z^2 = 4$, $x = 0$

2 pts 2 pts

B) (6 points) Write inequalities to describe the solid rectangular box in the first octant bounded by the planes $x = 4$, $y = 7$ and $z = 8$

$0 \le x \le 4$, $0 \le y \le 7$, $0 \le z \le 8$

2 pts 2 pts 2 pts

6. A) (6 points) Find the component form of the vector $\vec{v}$ making an angle of $\frac{\pi}{6}$ with the positive x-axis with $|\vec{v}| = 2$.

$$\vec{V} = \langle |\vec{v}| \cos(\tfrac{\pi}{6}), \ |\vec{v}| \sin(\tfrac{\pi}{6}) \rangle \quad \text{(3 pts)}$$

$$= \langle 2 \cdot \tfrac{\sqrt{3}}{2}, \ 2 \cdot \tfrac{1}{2} \rangle \quad \text{(2 pts)}$$

$$= \langle \sqrt{3}, 1 \rangle = \sqrt{3}\,\vec{i} + \vec{j} \quad \text{(1 pt)}$$

B) (6 points)Find the angle between the vectors $\vec{u} = \langle 1, -2, 3 \rangle$ and $\vec{v} = \langle -2, 3, 1 \rangle$.

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|\,|\vec{v}|} \quad \text{(1 pt)}$$

$$\vec{u} \cdot \vec{v} = (1)(-2) + (-2)(3) + (3)(1) = -5 \quad \text{(1 pt)}$$

$$|\vec{u}| = \sqrt{14} \ \text{(1 pt)}, \quad |\vec{v}| = \sqrt{14} \ \text{(1 pt)} \quad \text{so}$$

$$\cos\theta = \frac{-5}{14} \ \text{(2 pts)} \implies \theta = \cos^{-1}\left(\frac{-5}{14}\right)$$

7. A) (8 points) If the angle between $\vec{u}$ and $\vec{v}$ is $\theta = \dfrac{\pi}{3}$ and that $|\vec{u}| = 3$, $|\vec{v}| = 2$, find $|\vec{u} + \vec{v}|$.

$$|\vec{u}+\vec{v}|^2 = (\vec{u}+\vec{v}) \cdot (\vec{u}+\vec{v}) \quad \text{2pts}$$

$$= \vec{u}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v} \quad \text{2pts}$$

$$= |\vec{u}|^2 + 2\vec{u}\cdot\vec{v} + |\vec{v}|^2 \quad \text{1 pt}$$

$$= |\vec{u}|^2 + |\vec{v}|^2 + 2|\vec{u}||\vec{v}|\cos\frac{\pi}{3} \quad \text{2pts}$$

$$= 9 + 4 + 2(3)(2)\left(\frac{1}{2}\right)$$

$$= 19 \quad \text{1 pt}$$

$\Longrightarrow |\vec{u}+\vec{v}| = \sqrt{19}$

$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$

$3 \quad 2 \quad \frac{1}{2}$

3

8. A) (6 points) Determine whether the vectors $\vec{u} = <1, 4, -7>, \vec{v} = <2, -1, 4>$ and $\vec{w} = <0, -9, 18>$ lie on the same plane.

$$V = |\vec{u} \cdot (\vec{v} \times \vec{w})| = 0$$ 4 pts

We conclude that the three vectors are on the same plane. 2 pts

B) (8 points) Let $\vec{u} = <1, -1, 2>$, and $\vec{v} = <3, -2, 0>$. If $\vec{a} = \text{proj}_{\vec{v}}\vec{u}$,

fid find $w = (2\vec{u} - \vec{a}) \cdot \dfrac{\vec{v}}{3}$

We know that $\vec{u} - \text{proj}_{\vec{v}}\vec{u}$ is orthogonal to $\vec{v}$ so $\vec{u} - \vec{a}$ is orthogonal to $\vec{v}$.

$$w = (\vec{u} + \vec{u} - \vec{a}) \cdot \frac{\vec{v}}{3}$$ 2 pts

$$= \vec{u} \cdot \frac{\vec{v}}{3} + (\vec{u} - \vec{a}) \cdot \frac{\vec{v}}{3}$$ 1 pt

$$= \frac{1}{3}\vec{u} \cdot \vec{v} + 0$$ 4 pts

$$= \frac{1}{3}(3 + 2 + 0) = \frac{5}{3}$$ 1 pt