Q	MM	V1	V2	V3	V4
1	a	e	b	c	e
2	a	b	a	c	d
3	a	d	e	d	c
4	a	a	a	e	a
5	a	e	d	e	c
6	a	a	d	e	c
7	a	e	c	d	c
8	a	d	d	b	b
9	a	c	d	b	e
10	a	c	a	c	a
11	a	a	a	e	b
12	a	b	e	d	a
13	a	c	d	d	d
14	a	d	b	c	c
15	a	b	c	d	a
16	a	e	d	a	e
17	a	d	c	a	d
18	a	d	c	d	e
19	a	d	a	d	e
20	a	d	b	b	c
21	a	c	c	c	e
22	a	b	a	d	e
23	a	e	d	d	e
24	a	a	d	d	a
25	a	c	e	a	c
26	a	a	c	e	a
27	a	b	c	a	c
28	a	c	c	d	a

Detailed Solutions

King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

Math 101
Final Exam
Term 131
Sunday 29/12/2013
Net Time Allowed: 180 minutes

MASTER VERSION

1. An equation for the **tangent line** to the curve $y = \dfrac{x-1}{x+1}$ at $x = 0$ is

(a) $y = 2x - 1$

(b) $y = 3x - 1$

(c) $y = -x - 1$

(d) $y = 4x + 1$

(e) $y = 3x - 2$

• $x = 0 \Rightarrow y = -1 \Rightarrow (0, -1)$

• $y' = \dfrac{(x+1) - (x-1)}{(x+1)^2} = \dfrac{2}{(x+1)^2}$

Slope $= y'|_{x=0} = \dfrac{2}{(0+1)^2} = 2$

. Eq.

$y + 1 = 2(x - 0)$

$\Rightarrow y = 2x - 1$

2. $\lim\limits_{x \to -1^+} \dfrac{x}{\sqrt{x+1}} =$

(a) $-\infty$

(b) ∞

(c) 0

(d) 1

(e) 2

Num $\to -1$

Deno $\to 0^+$

So $\lim\limits_{x \to -1^+} \dfrac{x}{\sqrt{x+1}} = -\infty$

$(\to \frac{-1}{0})$

3. Where is $f(x) = \ln(1 - \sqrt{x})$ **continuous**? In its domain:

$\cdot\ \sqrt{x} \implies x \geq 0$

$\cdot\ \ln(1-\sqrt{x}) \implies 1-\sqrt{x} > 0 \implies \sqrt{x} < 1 \implies x < 1$

Combining, we get

$0 \leq x < 1$

(a) $[0, 1)$

(b) $(0, 1)$

(c) $(0, \infty)$

(d) $(1, \infty)$

(e) $(0, \infty)$

4. $\sum_{k=1}^{n} \left(\frac{1}{n} + 2k\right) =$

$$\sum_{k=1}^{n} \frac{1}{n} + \sum_{k=1}^{n} 2k$$

$$= \frac{1}{n}\sum_{k=1}^{n} 1 + 2\sum_{k=1}^{n} k$$

$$= \frac{1}{n} \cdot n + 2 \cdot \frac{n(n+1)}{2}$$

$$= 1 + n(n+1)$$

$$= n^2 + n + 1$$

(a) $n^2 + n + 1$

(b) $2n^2 + n + 3$

(c) $\frac{1}{n} + n^2$

(d) $\frac{1}{n} + n^2 + n$

(e) $n^2 + 2n + 2$

5. The slope of the **normal line** to the curve

$$y^3 + \tan(2x) = \cos(3xy)$$

when $x = 0$ is

$\cdot\ x=0 \Rightarrow y^3 + 0 = 1 \Rightarrow y = 1$

$\cdot\ 3y^2 y' + 2\sec^2(2x) = -\sin(3xy)\cdot 3[xy' + y]$

$\xRightarrow[y=1]{x=0}\ 3y' + 2 = -0\cdot 3(0\cdot y' + 1) = 0$

$\Rightarrow\ y' = -\frac{2}{3}$ (slope of tangent)

slope of normal is $-\frac{1}{-\frac{2}{3}} = \frac{3}{2}$

(a) $\frac{3}{2}$

(b) $-\frac{3}{2}$

(c) $\frac{2}{3}$

(d) $\frac{5}{2}$

(e) $-\frac{5}{3}$

6. If $\lim_{x\to 3} \frac{f(x)-4}{x-3} = 5$, then $\lim_{x\to 3} x\,f(x) =$

as the limit $\lim_{x\to 3} \frac{f(x)-4}{x-3} = 5$ exists, we must have $\lim_{x\to 3}(f(x)-4) = 0$ & so

$\lim_{x\to 3} f(x) = 4$

Thus, $\lim_{x\to 3} x f(x) = 3\cdot 4 = 12.$

(a) 12

(b) 15

(c) 0

(d) 5

(e) -4

7. If $f(x) = 10x^2 + \frac{1}{2}\tan x$, then $f^{(3)}(x) =$

(a) $\sec^2 x\,(\sec^2 x + 2\tan^2 x)$

(b) $20 + \sec^4 x + 2\sec x \tan x$

(c) $\sec^3 x + \tan^2 x$

(d) $\sec^4 x + 2\,\sec x \tan^2 x$

(e) $\sec^2 x(\sec^2 x - \tan x)$

$f'(x) = 20x + \frac{1}{2}\sec^2 x$

$f''(x) = 20 + \frac{1}{2}\cdot 2\sec x \cdot \sec x \tan x$

$= 20 + \sec^2 x \cdot \tan x$

$f'''(x) = 0 + \sec^2 x \cdot \sec^2 x + \tan x \cdot 2\sec x \cdot \sec x \tan x$

$= \sec^4 x + 2\sec^2 x \tan^2 x$

$= \sec^2 x\,(\sec^2 x + 2\tan^2 x)$

8. Let $H(x) = f(xg(x) - 2)$. If $f(4) = 6$, $f'(4) = -5$, $g(2) = 3$, and $g'(2) = -2$, then $H'(2) =$

(a) 5

(b) -6

(c) 10

(d) -15

(e) 2

$H'(x) = f'(xg(x) - 2)\cdot[x\,g'(x) + g(x)]$

$H'(2) = f'(2g(2) - 2)\cdot[2\,g'(2) + g(2)]$

$= f'(4)\cdot[2(-2) + 3]$

$= -5\,(-4+3)$

$= -5\,(-1)$

$= 5$

9. Using a suitable **linear approximation**, $(1.0002)^{500} \approx$

$f(x) = x^{500}$, $a = 1$

$f'(x) = 500\, x^{499} \Rightarrow f'(1) = 500$

$f(x) \approx f(1) + f'(1)(x-1)$, when x is near 1

$f(x) \approx 1 + 500(x-1)$

$f(1.0002) \approx 1 + 500(1.0002 - 1)$

$= 1 + 500\left(\frac{2}{10000}\right)$

$= 1 + \frac{1000}{10000}$

$= 1 + \frac{1}{10}$

$= 1.10$

(a) 1.10

(b) 1.01

(c) 1.02

(d) 0.002

(e) 0.003

10. Let $f(x) = \dfrac{x^3 + 2x^2 - x - 2}{x^3 - x}$. If R is the number of **removable** discontinuities of f and I is the number of **infinite** discontinuities of f, then

$f(x) = \dfrac{x^2(x+2) - (x+2)}{x(x^2-1)}$

$= \dfrac{(x+2)(x^2-1)}{x(x^2-1)}$

$= \dfrac{(x+2)(x-1)(x+1)}{x(x-1)(x+1)}$

$= \dfrac{x+2}{x}$, $x \neq \pm 1$

Remov. at $x = \pm 1$

Infinite at $x = 0$

(a) $R = 2$ and $I = 1$

(b) $R = 1$ and $I = 2$

(c) $R = 0$ and $I = 3$

(d) $R = 3$ and $I = 0$

(e) $R = 3$ and $I = 3$

11. If **Newton's Method** is used to estimate the x-coordinate of the point of intersection of the curves $y = \sin\left(x + \frac{\pi}{2}\right)$ and $y = \ln(2x+1)$ with $x_0 = 0$, then $x_1 =$

$f(x) = \ln(2x+1) - \sin(x + \frac{\pi}{2})$

(a) 0.5 $\quad f'(x) = \frac{2}{2x+1} - \cos(x + \frac{\pi}{2})$

(b) 0.3 $\quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

(c) 0.1 $\quad = 0 - \frac{f(0)}{f'(0)} = 0 - \frac{-1}{2}$

(d) -0.1 $\quad = \frac{1}{2}$

(e) -0.2

12. $\lim_{x\to\infty}\left(\sqrt{x^2+3x} - 3\sqrt{x^2-2x}\right) =$

(a) $-\infty$ $\quad \lim_{x\to\infty} |x| \left(\sqrt{1+\frac{3}{x}} - 3\sqrt{1-\frac{2}{x}}\right)$

(b) ∞ $\quad = \infty\,(1-3)$

(c) 0 $\quad = -\infty$

(d) -3

(e) 4

13. If $\frac{dy}{dx} = 2\cos x - \frac{1}{3}\sec x \tan x$ and $y(\pi) = \frac{1}{12}$, then $y(0) =$

(a) $-\frac{7}{12}$

(b) $\frac{1}{4}$

(c) $-\frac{1}{3}$

(d) $\frac{5}{12}$

(e) $-\frac{1}{12}$

• $y = 2\sin x - \frac{1}{3}\sec x + C$

• $y(\pi) = \frac{1}{12} \Rightarrow \frac{1}{12} = 2\sin(\pi) - \frac{1}{3}\sec(\pi) + C$

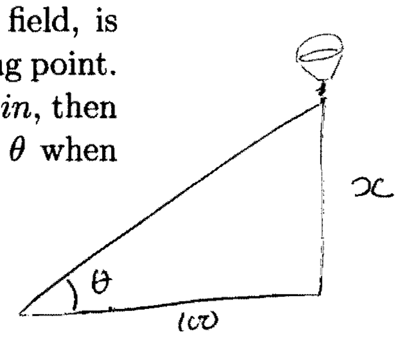
$\Rightarrow \frac{1}{12} = 0 + \frac{1}{3} + C$

$\Rightarrow C = \frac{1}{12} - \frac{1}{3} = \frac{-3}{12} = -\frac{1}{4}$

So $y = 2\sin x - \frac{1}{3}\sec x - \frac{1}{4}$

& $y(0) = 2(0) - \frac{1}{3}(1) - \frac{1}{4} = -\frac{1}{3} - \frac{1}{4} = -\frac{7}{12}$

14. A hot air balloon, rising straight up from a level field, is tracked by a boy $100\,m$ on the ground from the lifting point. If the balloon is rising at a constant rate of $50\,m/min$, then **the rate of change** of the boy's elevation angle θ when $\theta = \frac{\pi}{4}$ is



$\frac{dx}{dt} = 50\ m/min$

(a) 0.25 rad/min

(b) 0.20 rad/min

(c) 0.0125 rad/min

(d) 0.025 rad/min

(e) 0.125 rad/min

$\tan\theta = \frac{x}{100}$

$\Rightarrow x = 100\tan\theta$

• $\frac{dx}{dt} = 100\sec^2\theta\,\frac{d\theta}{dt}$

$\Rightarrow 50 = 100(\sqrt{2})^2\,\frac{d\theta}{dt}$

$\Rightarrow \frac{d\theta}{dt} = \frac{50}{200}$

$= \frac{1}{4} = 0.25$

15. $\lim_{x\to\infty}(1+2x)^{\frac{1}{2\ln x}} =$

(a) $\sqrt{e}$

(b) e^2

(c) $\frac{1}{\sqrt{e}}$

(d) e

(e) 1

$y = (1+2x)^{\frac{1}{2\ln x}}$

$\ln y = \frac{1}{2\ln x}\ln(1+2x) = \frac{\ln(1+2x)}{2\ln x}$

$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(1+2x)}{2\ln x} \quad \frac{\infty}{\infty}$

$\stackrel{L'R}{=} \lim_{x\to\infty} \frac{\frac{2}{1+2x}}{\frac{2}{x}}$

$= \lim_{x\to\infty} \frac{x}{2x+1}$

$= \frac{1}{2}$

So $\lim_{x\to\infty} y = e^{1/2} = \sqrt{e}$

16. The **absolute maximum value** of $f(x) = 2\sin(2x) + \cos(4x)$ on the interval $\left[0, \frac{\pi}{6}\right]$ is equal to

(a) $\frac{3}{2}$

(b) 1

(c) 3

(d) $\frac{\sqrt{3}}{2}$

(e) $\frac{1+\sqrt{3}}{2}$

$f'(x) = 4\cos(2x) - 4\sin(4x)$

$= 4\cos(2x) - 8\sin(2x)\cos(2x)$

$= 4\cos(2x)\left[1 - 2\sin(2x)\right]$

$f'(x) = 0 \Rightarrow \cos(2x) = 0, \quad \sin(2x) = \frac{1}{2}$

$\Rightarrow 2x = \frac{\pi}{2}, \quad 2x = \frac{\pi}{6}$

$\Rightarrow x = \frac{\pi}{4}, \quad x = \frac{\pi}{12}$

$\Rightarrow x = \frac{\pi}{12} \in \left[0, \frac{\pi}{6}\right]$

• $f\left(\frac{\pi}{12}\right) = 2\sin\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{3}\right) = 2\left(\frac{1}{2}\right) + \frac{1}{2} = 1 + \frac{1}{2} = \frac{3}{2}$ ✓

• $f(0) = 2(0) + 1 = 1$

• $f\left(\frac{\pi}{6}\right) = 2\sin\left(\frac{\pi}{3}\right) + \cos\left(\frac{2\pi}{3}\right) = 2\cdot\frac{\sqrt{3}}{2} - \frac{1}{2}$

$= \sqrt{3} - \frac{1}{2}$

17. If $y = \dfrac{(x+10)^{x+10}}{x^x}$, then $y' =$

(a) $y \ln\left(1 + \dfrac{10}{x}\right)$

(b) $y \ln 10$

(c) $yx \ln(x+10)$

(d) $yx(x+10)$

(e) $\dfrac{xy}{\ln(x+10)}$

$$\ln y = (x+10)\ln(x+10) - x\ln x$$

$$\frac{1}{y}\cdot y' = (x+10)\cdot\frac{1}{x+10} + \ln(x+10) - \left(x\cdot\frac{1}{x} + \ln x\right)$$

$$= 1 + \ln(x+10) - 1 - \ln x$$

$$= \ln\left(\frac{x+10}{x}\right)$$

$$\Longrightarrow y' = y\ln\left(1+\frac{10}{x}\right)$$

18. The graph of $f(x) = \dfrac{e^x}{e^x + 1}$

(a) is concave up on $(-\infty, 0)$

(b) is concave up on $(0, \infty)$

(c) has two inflection points

(d) has no inflection points

(e) is concave up on $(-\infty, \infty)$

$$f'(x) = \frac{(e^x+1)e^x - e^x(e^x)}{(e^x+1)^2} = \frac{e^x}{(e^x+1)^2}$$

$$f''(x) = \frac{(e^x+1)^2 e^x - e^x\cdot 2(e^x+1)e^x}{(e^x+1)^4}$$

$$= \frac{(e^x+1)e^x - 2e^{2x}}{(e^x+1)^3}$$

$$= \frac{e^x - e^{2x}}{(e^x+1)^3}$$

$$= \frac{e^x(1 - e^x)}{(e^x+1)^3}$$

- $f''(x) = 0 \Longrightarrow 1 - e^x = 0 \Longrightarrow x = 0$
- $f''(x)$ exists for all x

f'': $+$ up | 0 | $-$ down

↑ one inflection pt

19. Let $f(x) = \alpha x^2 + \beta x + \gamma$, where $\alpha \neq 0, \beta, \gamma$ are constants. The value of c that satisfies the conclusion of the **Mean Value Theorem** for f on the interval $[3, 7]$ is

$f'(x) = 2\alpha x + \beta$

$f'(c) = \dfrac{f(7) - f(3)}{7-3}$

$2\alpha c + \beta = \dfrac{(49\alpha + 7\beta + \gamma) - (9\alpha + 3\beta + \gamma)}{4}$

$8\alpha c + 4\beta = 40\alpha + 4\beta$

$8\alpha c = 40\alpha$

$c = \dfrac{40\alpha}{8\alpha}$

$= 5 \quad , \alpha \neq 0$

(a) 5

(b) 6

(c) 3.5

(d) 4.5

(e) 4

20. The **number** of critical points of $f(x) = (x - x^3)^{-1/3}$ is

$f'(x) = -\frac{1}{3}(x - x^3)^{-4/3} \cdot (1 - 3x^2)$

$f'(x) = 0 \Rightarrow 1 - 3x^2 = 0 \Rightarrow x = \pm \frac{1}{\sqrt{3}} \in$ domain of f

$f'(x)$ DNE if $x - x^3 = 0 \Rightarrow x(1 - x^2) = 0$
$\Rightarrow x = 0, \pm 1 \notin$ domain of f

Critical points at $x = \pm \frac{1}{\sqrt{3}}$

(a) 2

(b) 3

(c) 4

(d) 5

(e) 1

21. The function $f(x) = x^4 - 4x^3 + 4x^2 + 4$ has

(a) a local max at $x = 1$ and a local min at $x = 0$ and $x = 2$

(b) a local max at $x = 0$ and a local min at $x = 1$

(c) a local max at $x = 2$ and a local min at $x = -1$

(d) a local max at $x = 2$ and a local min at $x = 0$ and $x = 1$

(e) a local max at $x = 1$ and a local min at $x = -1$ and $x = 0$

$f'(x) = 4x^3 - 12x^2 + 8x$
$= 4x(x^2 - 3x + 2)$
$= 4x(x-1)(x-2)$
$f'(x) = 0 \Rightarrow x = 0, 1, 2$

$-$ $+$ $-$ $+$
0 1 2
min max min

22. The function $g(x) = e^{-\frac{1}{x}}$ is

(a) increasing on $(-\infty, 0)$ and $(0, \infty)$

(b) increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$

(c) decreasing on $(0, \infty)$

(d) increasing on $(0, 1)$ and decreasing on $(1, \infty)$

(e) decreasing on $(-\infty, 0)$ and $(0, \infty)$

$g'(x) = e^{-1/x} \cdot \frac{1}{x^2} > 0$ for all $x \neq 0$

g' $+$ 0 $+$

23. If $y = \tan^{-1}\left(\dfrac{1-\cot x}{1+\cot x}\right)$, then $y' =$ $\dfrac{1}{1+\left(\frac{1-\cot x}{1+\cot x}\right)^2} \cdot \dfrac{(1+\cot x)\csc^2 x - (1-\cot x)(-\csc^2 x)}{(1+\cot x)^2}$

(a) 1

(b) 0

(c) 2

(d) $\csc^2 x$

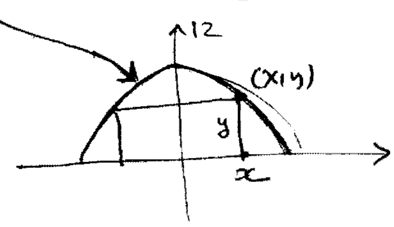
(e) $\dfrac{2}{1+\csc^2 x}$

$$= \frac{2\csc^2 x}{(1+\cot x)^2 + (1-\cot x)^2}$$

$$= \frac{2\csc^2 x}{1 + 2\cot x + \cot^2 x + 1 - 2\cot x + \cot^2 x}$$

$$= \frac{2\csc^2 x}{2(1+\cot^2 x)} = \frac{2\csc^2 x}{2\csc^2 x} = \underline{1}$$

24. A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the **largest area** that the rectangle can have?



(a) 32

(b) 30

(c) $20\sqrt{2}$

(d) $8\sqrt{6}$

(e) 28

$A = 2xy$

$A(x) = 2x(12-x^2)$

$= 24x - 2x^3, \quad 0 \le x \le \sqrt{12}$

$A'(x) = 24 - 6x^2$

$A'(x) = 0 \Rightarrow x = 2 \quad (x \ge 0)$

$A''(x) = -12x \ ; \ A''(2) = -24 < 0,$ max

$A(2) = 2(2)(12-4)$

$= \underline{32}$

25. $\int \frac{1}{x}\left(\frac{2}{x}+\frac{x}{2}\right)^2 dx =$ $\int \frac{1}{x}\left(\frac{4}{x^2}+2+\frac{x^2}{4}\right) dx$

$= \int \left(4x^{-3}+\frac{2}{x}+\frac{x}{4}\right) dx$

$= 4\,\frac{x^{-2}}{-2} + 2\ln|x| + \frac{1}{4}\,\frac{x^2}{2} + C$

$= -\frac{2}{x^2} + 2\ln|x| + \frac{x^2}{8} + C$

(a) $-\frac{2}{x^2}+\frac{x^2}{8}+2\ln|x|+C$

(b) $-\frac{2}{x^2}+\frac{x^2}{8}+C$

(c) $\left(-\frac{2}{x^2}+\frac{x^2}{8}\right)\ln|x|+C$

(d) $-\frac{2}{x^2}-2\ln|x|+\frac{x}{4}+C$

(e) $\frac{2}{x^2}+(\ln|x|)^{-1}+\frac{x^2}{4}+C$

26. Using 100 rectangles of equal width and **right endpoints**, the area below the graph of $f(x)=4x^3$ and above interval $[0,1]$ is approximately equal to

- $\Delta x = \frac{1-0}{n} = \frac{1}{n}$; $x_k = 0 + \frac{1}{n}k = \frac{k}{n}$
- $A_n = \sum_{k=1}^{n} f(x_k)\,\Delta k$
  $= \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\cdot\frac{1}{n} = \sum_{k=1}^{n} 4\left(\frac{k}{n}\right)^3\cdot\frac{1}{n}$
  $= \frac{4}{n^4}\sum_{k=1}^{n} k^3 = \frac{4}{n^4}\left[\frac{n(n+1)}{2}\right]^2$
  $= \frac{(n+1)^2}{n^2} = \left(1+\frac{1}{n}\right)^2$
- $A_{100} = \left(1+\frac{1}{100}\right)^2 = 1.01^2$
  $= 1.0201$

(a) 1.0201

(b) 1.01

(c) 1.03

(d) 1.0112

(e) 1.00

27. If the curves $y = x^2 + bx + a$ and $y = cx - x^2$ have a **common tangent line** at the point $(1,1)$, then $a+2b+c =$

• $(1,1)$ is on both curves:

$1 = 1 + b + a \qquad a + b = 0$

$1 = c - 1 \quad \Rightarrow \quad c = 2$

• slopes on both curves at $(1,1)$ are equal:

$2x + b\big|_{x=1} = c - 2x\big|_{x=1}$

$2 + b = c - 2 \quad \Rightarrow \quad 2 + b = 2 - 2$

$\Rightarrow b = -2$

$\Rightarrow a = 2$

• So $a + 2(b) + c$

$= 2 - 4 + 2$

$= 0$

(a) 0

(b) 2

(c) 4

(d) -2

(e) -3

28. If α is a positive real number and

$$\lim_{x\to 0} \frac{\sin^\alpha(8x)}{\ln^\alpha(1+4x)} = 64,$$

then $\alpha =$

$\lim_{x\to 0} \left(\frac{\sin(8x)}{\ln(1+4x)}\right)^\alpha = 64 \quad (*)$

as $\lim \frac{\sin(8x)}{\ln(1+4x)} \overset{L'R}{=} \lim_{x\to 0} \frac{8\cos(8x)}{\frac{4}{1+4x}} = \frac{8 \cdot 1}{\frac{4}{1}} = 2$

then from $(*)$, we get

$2^\alpha = 64$

So $\alpha = 6$

(a) 6

(b) 8

(c) 4

(d) 7

(e) 9