

Q	MM	V1	V2	V3	V4
1	a	e	b	c	e
2	a	b	a	c	d
3	a	d	e	d	c
4	a	a	a	e	a
5	a	e	d	e	c
6	a	a	d	e	c
7	a	e	c	d	c
8	a	d	d	b	b
9	a	c	d	b	e
10	a	c	a	c	a
11	a	a	a	e	b
12	a	b	e	d	a
13	a	c	d	d	d
14	a	d	b	c	c
15	a	b	c	d	a
16	a	e	d	a	e
17	a	d	c	a	d
18	a	d	c	d	e
19	a	d	a	d	e
20	a	d	b	b	c
21	a	c	c	c	e
22	a	b	a	d	e
23	a	e	d	d	e
24	a	a	d	d	a
25	a	c	e	a	c
26	a	a	c	e	a
27	a	b	c	a	c
28	a	c	c	d	a

Detailed
Solutions →

King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

Math 101
Final Exam
Term 131
Sunday 29/12/2013
Net Time Allowed: 180 minutes

MASTER VERSION

1. An equation for the **tangent line** to the curve $y = \frac{x-1}{x+1}$ at $x = 0$ is

(a) $y = 2x - 1$

(b) $y = 3x - 1$

(c) $y = -x - 1$

(d) $y = 4x + 1$

(e) $y = 3x - 2$

$$\bullet x=0 \Rightarrow y=-1 \Rightarrow (0, -1)$$

$$\bullet y' = \frac{(x+1) - (x-1)}{(x+1)^2} = \frac{2}{(x+1)^2}$$

$$\text{Slope} = y'|_{x=0} = \frac{2}{(0+1)^2} = 2$$

• Eq.

$$y+1 = 2(x-0)$$

$$\Rightarrow y = 2x - 1$$

2. $\lim_{x \rightarrow -1^+} \frac{x}{\sqrt{x+1}} =$

(a) $-\infty$

(b) ∞

(c) 0

(d) 1

(e) 2

$$\text{Num} \rightarrow -1$$

$$\text{Deno} \rightarrow 0^+$$

$$\text{So } \lim_{x \rightarrow -1^+} \frac{x}{\sqrt{x+1}} = -\infty$$

$$\left(\rightarrow \frac{-1}{0} \right)$$

3. Where is $f(x) = \ln(1 - \sqrt{x})$ continuous? In its domain:

(a) $[0, 1)$

(b) $(0, 1)$

(c) $(0, \infty)$

(d) $(1, \infty)$

(e) $(0, \infty)$

$$\bullet \sqrt{x} \Rightarrow x \geq 0$$

$$\bullet \ln(1 - \sqrt{x}) \Rightarrow 1 - \sqrt{x} > 0 \Rightarrow \sqrt{x} < 1 \Rightarrow x < 1$$

Combining, we get

$$0 \leq x < 1$$

4. $\sum_{k=1}^n \left(\frac{1}{n} + 2k \right) =$

$$\sum_{k=1}^n \frac{1}{n} + \sum_{k=1}^n 2k$$

$$= \frac{1}{n} \sum_{k=1}^n 1 + 2 \sum_{k=1}^n k$$

$$= \frac{1}{n} \cdot n + 2 \cdot \frac{n(n+1)}{2}$$

$$= 1 + n(n+1)$$

$$= n^2 + n + 1$$

(a) $n^2 + n + 1$

(b) $2n^2 + n + 3$

(c) $\frac{1}{n} + n^2$

(d) $\frac{1}{n} + n^2 + n$

(e) $n^2 + 2n + 2$

5. The slope of the **normal line** to the curve

$$y^3 + \tan(2x) = \cos(3xy)$$

when $x = 0$ is

- $x=0 \Rightarrow y^3 + 0 = 1 \Rightarrow y=1$
 $\cdot 3y^2 y' + 2 \sec^2(2x) = -\sin(3xy) \cdot 3[xy' + y]$
 $\xrightarrow[x=y=1]{x=0} 3y' + 2 = -0 \cdot 3(0 \cdot y' + 1) = 0$
 $\Rightarrow y' = -\frac{2}{3}$ (slope of tangent)
 slope of normal is $-\frac{1}{-\frac{2}{3}} = \frac{3}{2}$
- (a) $\frac{3}{2}$
- (b) $-\frac{3}{2}$
- (c) $\frac{2}{3}$
- (d) $\frac{5}{2}$
- (e) $-\frac{5}{3}$

6. If $\lim_{x \rightarrow 3} \frac{f(x) - 4}{x - 3} = 5$, then $\lim_{x \rightarrow 3} x f(x) =$

- (a) 12
- (b) 15
- (c) 0
- (d) 5
- (e) -4

as the limit $\lim_{x \rightarrow 3} \frac{f(x) - 4}{x - 3} = 5$ exists, we must have $\lim_{x \rightarrow 3} (f(x) - 4) = 0$ & so $\lim_{x \rightarrow 3} f(x) = 4$

Thus, $\lim_{x \rightarrow 3} x f(x) = 3 \cdot 4 = 12.$

7. If $f(x) = 10x^2 + \frac{1}{2} \tan x$, then $f^{(3)}(x) =$

(a) $\sec^2 x (\sec^2 x + 2 \tan^2 x)$

(b) $20 + \sec^4 x + 2 \sec x \tan x$

(c) $\sec^3 x + \tan^2 x$

(d) $\sec^4 x + 2 \sec x \tan^2 x$

(e) $\sec^2 x (\sec^2 x - \tan x)$

$$f'(x) = 20x + \frac{1}{2} \sec^2 x$$

$$f''(x) = 20 + \frac{1}{2} \cdot 2 \sec x \cdot \sec x \tan x$$

$$= 20 + \sec^2 x \cdot \tan x$$

$$f'''(x) = 0 + \sec^2 x \cdot \sec^2 x + \tan x \cdot 2 \sec x \cdot \sec x \tan x$$

$$= \sec^4 x + 2 \sec^2 x \tan^2 x$$

$$= \sec^2 x (\sec^2 x + 2 \tan^2 x)$$

8. Let $H(x) = f(xg(x) - 2)$. If $f(4) = 6$, $f'(4) = -5$, $g(2) = 3$, and $g'(2) = -2$, then $H'(2) =$

(a) 5

(b) -6

(c) 10

(d) -15

(e) 2

$$H'(x) = f'(xg(x) - 2) \cdot [xg'(x) + g(x)]$$

$$H'(2) = f'(2g(2) - 2) \cdot [2g'(2) + g(2)]$$

$$= f'(4) \cdot [2(-2) + 3]$$

$$= -5(-4 + 3)$$

$$= -5(-1)$$

$$= 5$$

9. Using a suitable **linear approximation**, $(1.0002)^{500} \approx$

(a) 1.10

(b) 1.01

(c) 1.02

(d) 0.002

(e) 0.003

$$f(x) = x^{500}, \quad a=1$$

$$f'(x) = 500x^{499} \Rightarrow f'(1) = 500$$

$$f(x) \approx f(1) + f'(1)(x-1), \quad \text{when } x \text{ is near } 1$$

$$f(x) \approx 1 + 500(x-1)$$

$$f(1.0002) \approx 1 + 500(1.0002 - 1)$$

$$= 1 + 500\left(\frac{2}{10000}\right)$$

$$= 1 + \frac{1000}{10000}$$

$$= 1 + \frac{1}{10}$$

$$= 1.10$$

10. Let $f(x) = \frac{x^3 + 2x^2 - x - 2}{x^3 - x}$. If R is the number of **removable** discontinuities of f and I is the number of **infinite** discontinuities of f , then

- (a) $R = 2$ and $I = 1$
- (b) $R = 1$ and $I = 2$
- (c) $R = 0$ and $I = 3$
- (d) $R = 3$ and $I = 0$
- (e) $R = 3$ and $I = 3$

$$f(x) = \frac{x^2(x+2) - (x+2)}{x(x^2-1)}$$

$$= \frac{(x+2)(x^2-1)}{x(x^2-1)}$$

$$= \frac{(x+2)(x-1)(x+1)}{x(x-1)(x+1)}$$

$$= \frac{x+2}{x}, \quad x \neq \pm 1$$

Remov. at $x = \pm 1$
 Infinite at $x = 0$

11. If **Newton's Method** is used to estimate the x -coordinate of the point of intersection of the curves $y = \sin\left(x + \frac{\pi}{2}\right)$ and $y = \ln(2x + 1)$ with $x_0 = 0$, then $x_1 =$

$$f(x) = \ln(2x+1) - \sin\left(x + \frac{\pi}{2}\right)$$

$$(a) \quad 0.5 \quad f'(x) = \frac{2}{2x+1} - \cos\left(x + \frac{\pi}{2}\right)$$

$$(b) \quad 0.3 \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$(c) \quad 0.1 \quad = 0 - \frac{f(0)}{f'(0)} = 0 - \frac{-1}{2}$$

$$(d) \quad -0.1 \quad = \frac{1}{2}$$

$$(e) \quad -0.2$$

12. $\lim_{x \rightarrow \infty} (\sqrt{x^2 + 3x} - 3\sqrt{x^2 - 2x}) =$

$$(a) \quad -\infty \quad \lim_{x \rightarrow \infty} |x| \left(\sqrt{1 + \frac{3}{x}} - 3\sqrt{1 - \frac{2}{x}} \right)$$

$$(b) \quad \infty \quad = \infty (1 - 3)$$

$$(c) \quad 0 \quad = -\infty$$

$$(d) \quad -3$$

$$(e) \quad 4$$

13. If $\frac{dy}{dx} = 2 \cos x - \frac{1}{3} \sec x \tan x$ and $y(\pi) = \frac{1}{12}$, then $y(0) =$

(a) $-\frac{7}{12}$

(b) $\frac{1}{4}$

(c) $-\frac{1}{3}$

(d) $\frac{5}{12}$

(e) $-\frac{1}{12}$

• $y = 2 \sin x - \frac{1}{3} \sec x + C$

• $y(\pi) = \frac{1}{12} \Rightarrow \frac{1}{12} = 2 \sin(\pi) - \frac{1}{3} \sec(\pi) + C$

$\Rightarrow \frac{1}{12} = 0 + \frac{1}{3} + C$

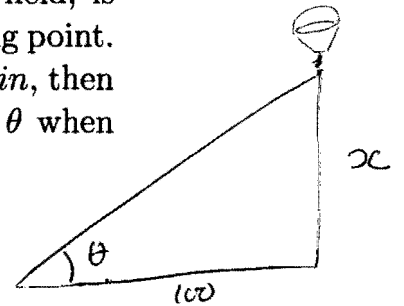
$\Rightarrow C = \frac{1}{12} - \frac{1}{3} = \frac{-3}{12} = -\frac{1}{4}$

So $y = 2 \sin x - \frac{1}{3} \sec x - \frac{1}{4}$

• $y(0) = 2(0) - \frac{1}{3}(1) - \frac{1}{4} = -\frac{1}{3} - \frac{1}{4} = -\frac{7}{12}$

14. A hot air balloon, rising straight up from a level field, is tracked by a boy 100 m on the ground from the lifting point. If the balloon is rising at a constant rate of 50 m/min, then **the rate of change** of the boy's elevation angle θ when $\theta = \frac{\pi}{4}$ is

$$\frac{dx}{dt} = 50 \text{ m/min}$$



- (a) 0.25 rad/min
- (b) 0.20 rad/min
- (c) 0.0125 rad/min
- (d) 0.025 rad/min
- (e) 0.125 rad/min

$$\tan \theta = \frac{x}{100}$$

$$\Rightarrow x = 100 \tan \theta$$

• $\frac{dx}{dt} = 100 \sec^2 \theta \frac{d\theta}{dt}$

$$\Rightarrow 50 = 100 (\sqrt{2})^2 \frac{d\theta}{dt}$$

$$\Rightarrow \frac{d\theta}{dt} = \frac{50}{200}$$

$$= \frac{1}{4} = 0.25$$

15. $\lim_{x \rightarrow \infty} (1+2x)^{\frac{1}{2 \ln x}} =$

(a) \sqrt{e}
 (b) e^2
 (c) $\frac{1}{\sqrt{e}}$
 (d) e
 (e) 1

$y = (1+2x)^{\frac{1}{2 \ln x}}$
 $\ln y = \frac{1}{2 \ln x} \ln(1+2x) = \frac{\ln(1+2x)}{2 \ln x}$
 $\lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} \frac{\ln(1+2x)}{2 \ln x} \quad \frac{\infty}{\infty}$
 $\stackrel{L'H}{=} \lim_{x \rightarrow \infty} \frac{\frac{2}{1+2x}}{\frac{2}{x}}$
 $= \lim_{x \rightarrow \infty} \frac{x}{2x+1}$
 $= \frac{1}{2}$
 So $\lim_{x \rightarrow \infty} y = e^{1/2} = \sqrt{e}$

16. The absolute maximum value of $f(x) = 2 \sin(2x) + \cos(4x)$ on the interval $\left[0, \frac{\pi}{6}\right]$ is equal to

(a) $\frac{3}{2}$
 (b) 1
 (c) 3
 (d) $\frac{\sqrt{3}}{2}$
 (e) $\frac{1+\sqrt{3}}{2}$

$f'(x) = 4 \cos(2x) - 4 \sin(4x)$
 $= 4 \cos(2x) - 8 \sin(2x) \cos(2x)$
 $= 4 \cos(2x) [1 - 2 \sin(2x)]$
 $f'(x) = 0 \Rightarrow \cos(2x) = 0, \sin(2x) = \frac{1}{2}$
 $\Rightarrow 2x = \frac{\pi}{2}, 2x = \frac{\pi}{6}$
 $\Rightarrow x = \frac{\pi}{4}, x = \frac{\pi}{12}$
 $\Rightarrow x = \frac{\pi}{12} \in \left[0, \frac{\pi}{6}\right]$

$\bullet f\left(\frac{\pi}{12}\right) = 2 \sin\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{3}\right) = 2\left(\frac{1}{2}\right) + \frac{1}{2} = 1 + \frac{1}{2} = \frac{3}{2} \checkmark$
 $\bullet f(0) = 2(0) + 1 = 1$
 $\bullet f\left(\frac{\pi}{6}\right) = 2 \sin\left(\frac{\pi}{3}\right) + \cos\left(\frac{2\pi}{3}\right) = 2 \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \sqrt{3} + \frac{\sqrt{3}}{2}$

17. If $y = \frac{(x+10)^{x+10}}{x^x}$, then $y' =$

(a) $y \ln\left(1 + \frac{10}{x}\right)$

(b) $y \ln 10$

(c) $yx \ln(x+10)$

(d) $yx(x+10)$

(e) $\frac{xy}{\ln(x+10)}$

$$\begin{aligned} \ln y &= (x+10) \ln(x+10) - x \ln x \\ \frac{1}{y} \cdot y' &= (x+10) \cdot \frac{1}{x+10} + \ln(x+10) - \left(x \cdot \frac{1}{x} + \ln x\right) \\ &= 1 + \ln(x+10) - 1 - \ln x \\ &= \ln\left(\frac{x+10}{x}\right) \\ \Rightarrow y' &= y \ln\left(1 + \frac{10}{x}\right) \end{aligned}$$

18. The graph of $f(x) = \frac{e^x}{e^x + 1}$

(a) is concave up on $(-\infty, 0)$

(b) is concave up on $(0, \infty)$

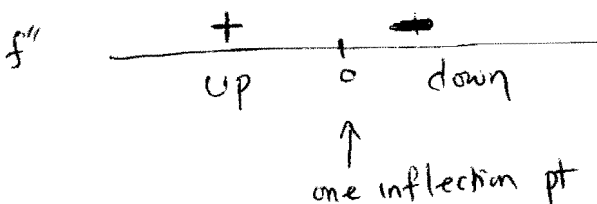
(c) has two inflection points

(d) has no inflection points

(e) is concave up on $(-\infty, \infty)$

$$\begin{aligned} f'(x) &= \frac{(e^x+1)e^x - e^x(e^x)}{(e^x+1)^2} = \frac{e^x}{(e^x+1)^2} \\ f''(x) &= \frac{(e^x+1)^2 e^x - e^x \cdot 2(e^x+1)e^x}{(e^x+1)^4} \\ &= \frac{(e^x+1)e^x - 2e^{2x}}{(e^x+1)^3} \\ &= \frac{e^x - e^{2x}}{(e^x+1)^3} \\ &= \frac{e^x(e^x - e^{2x})}{(e^x+1)^3} \end{aligned}$$

• $f''(x) = 0 \Rightarrow 1 - e^x = 0 \Rightarrow x = 0$
 • $f''(x)$ exists for all x



19. Let $f(x) = \alpha x^2 + \beta x + \gamma$, where $\alpha \neq 0, \beta, \gamma$ are constants. The value of c that satisfies the conclusion of the **Mean Value Theorem** for f on the interval $[3, 7]$ is

$$f'(x) = 2\alpha x + \beta$$

(a) 5

$$f'(c) = \frac{f(7) - f(3)}{7 - 3}$$

(b) 6

$$2\alpha c + \beta = \frac{(49\alpha + 7\beta + \gamma) - (9\alpha + 3\beta + \gamma)}{4}$$

(c) 3.5

$$8\alpha c + 4\beta = 40\alpha + 4\beta$$

(d) 4.5

$$8\alpha c = 40\alpha$$

(e) 4

$$c = \frac{40\alpha}{8\alpha} = 5, \alpha \neq 0$$

20. The number of critical points of $f(x) = (x - x^3)^{-1/3}$ is

$$f'(x) = -\frac{1}{3}(x - x^3)^{-4/3} \cdot (1 - 3x^2)$$

(a) 2

$$f'(x) = 0 \Rightarrow 1 - 3x^2 = 0 \Rightarrow x = \pm \frac{1}{\sqrt{3}} \in \text{domain of } f$$

(b) 3

$$f'(x) \text{ DNE } \text{ if } x - x^3 = 0 \Rightarrow x(1 - x^2) = 0$$

$$\Rightarrow x = 0, \pm 1 \notin \text{domain of } f$$

(c) 4

(d) 5

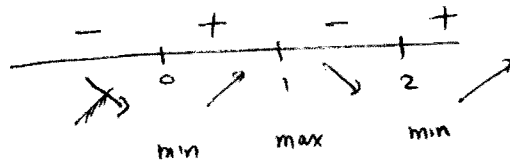
Critical points at $x = \pm \frac{1}{\sqrt{3}}$

(e) 1

21. The function $f(x) = x^4 - 4x^3 + 4x^2 + 4$ has

- (a) a local max at $x = 1$ and a local min at $x = 0$ and $x = 2$
- (b) a local max at $x = 0$ and a local min at $x = 1$
- (c) a local max at $x = 2$ and a local min at $x = -1$
- (d) a local max at $x = 2$ and a local min at $x = 0$ and $x = 1$
- (e) a local max at $x = 1$ and a local min at $x = -1$ and $x = 0$

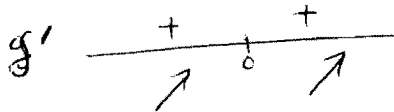
$$\begin{aligned} f'(x) &= 4x^3 - 12x^2 + 8x \\ &= 4x(x^2 - 3x + 2) \\ &= 4x(x-1)(x-2) \\ f'(x) = 0 &\Rightarrow x = 0, 1, 2 \end{aligned}$$



22. The function $g(x) = e^{-\frac{1}{x}}$ is

- (a) increasing on $(-\infty, 0)$ and $(0, \infty)$
- (b) increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$
- (c) decreasing on $(0, \infty)$
- (d) increasing on $(0, 1)$ and decreasing on $(1, \infty)$
- (e) decreasing on $(-\infty, 0)$ and $(0, \infty)$

$$g'(x) = e^{-\frac{1}{x}} \cdot \frac{1}{x^2} > 0 \text{ for all } x \neq 0$$



23. If $y = \tan^{-1} \left(\frac{1 - \cot x}{1 + \cot x} \right)$, then $y' = \frac{1}{1 + \left(\frac{1 - \cot x}{1 + \cot x} \right)^2} \cdot \frac{(1 + \cot x) \csc^2 x - (1 - \cot x) (-\csc^2 x)}{(1 + \cot x)^2}$

(a) 1

(b) 0

(c) 2

(d) $\csc^2 x$

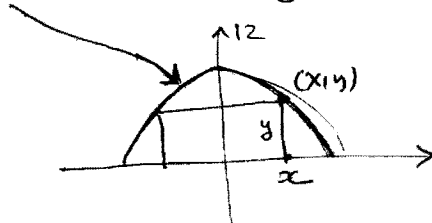
(e) $\frac{2}{1 + \csc^2 x}$

$$= \frac{2 \csc^2 x}{(1 + \cot x)^2 + (1 - \cot x)^2}$$

$$= \frac{2 \csc^2 x}{1 + 2 \cot x + \cot^2 x + 1 - 2 \cot x + \cot^2 x}$$

$$= \frac{2 \csc^2 x}{2(1 + \cot^2 x)} = \frac{2 \csc^2 x}{2 \csc^2 x} = 1$$

24. A rectangle has its base on the x -axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area that the rectangle can have?



(a) 32

(b) 30

(c) $20\sqrt{2}$

(d) $8\sqrt{6}$

(e) 28

$$A = 2xy$$

$$A(x) = 2x(12 - x^2)$$

$$= 24x - 2x^3, \quad 0 \leq x \leq \sqrt{12}$$

$$A'(x) = 24 - 6x^2$$

$$A'(x) = 0 \Rightarrow x = 2 \quad (x \geq 0)$$

$$A''(x) = -12x; \quad A''(2) = -24 < 0, \quad \underline{\underline{\text{max}}}$$

$$A(2) = 2(2)(12 - 4)$$

$$= 32$$

25. $\int \frac{1}{x} \left(\frac{2}{x} + \frac{x}{2} \right)^2 dx = \int \frac{1}{x} \left(\frac{4}{x^2} + 2 + \frac{x^2}{4} \right) dx$

$$= \int \left(4x^{-3} + \frac{2}{x} + \frac{x}{4} \right) dx$$

$$= 4 \frac{x^{-2}}{-2} + 2 \ln|x| + \frac{1}{4} \frac{x^2}{2} + C$$

$$= -\frac{2}{x^2} + 2 \ln|x| + \frac{x^2}{8} + C$$

(a) $-\frac{2}{x^2} + \frac{x^2}{8} + 2 \ln|x| + C$

(b) $-\frac{2}{x^2} + \frac{x^2}{8} + C$

(c) $\left(-\frac{2}{x^2} + \frac{x^2}{8} \right) \ln|x| + C$

(d) $-\frac{2}{x^2} - 2 \ln|x| + \frac{x}{4} + C$

(e) $\frac{2}{x^2} + (\ln|x|)^{-1} + \frac{x^2}{4} + C$

26. Using 100 rectangles of equal width and **right endpoints**, the area below the graph of $f(x) = 4x^3$ and above interval $[0, 1]$ is approximately equal to

$\Delta x = \frac{1-0}{n} = \frac{1}{n}$; $x_k = 0 + \frac{1}{n}k = \frac{k}{n}$

(a) 1.0201

(b) 1.01

(c) 1.03

(d) 1.0112

(e) 1.00

$$A_n = \sum_{k=1}^n f(x_k) \Delta x$$

$$= \sum_{k=1}^n f\left(\frac{k}{n}\right) \cdot \frac{1}{n} = \sum_{k=1}^n 4 \left(\frac{k}{n}\right)^3 \cdot \frac{1}{n}$$

$$= \frac{4}{n^4} \sum_{k=1}^n k^3 = \frac{4}{n^4} \left[\frac{n(n+1)}{2} \right]^2$$

$$= \frac{(n+1)^2}{n^2} = \left(1 + \frac{1}{n} \right)^2$$

$$A_{100} = \left(1 + \frac{1}{100} \right)^2 = 1.01^2 = 1.0201$$

27. If the curves $y = x^2 + bx + a$ and $y = cx - x^2$ have a common tangent line at the point $(1, 1)$, then $a + 2b + c =$

• $(1, 1)$ is on both curves:

$$\begin{aligned} 1 &= 1 + b + a & a + b &= 0 \\ 1 &= c - 1 & \Rightarrow c &= 2 \end{aligned}$$

(a) ~~0~~

(b) ~~2~~

(c) 4

(d) -2

(e) -3

• Slopes on both curves at $(1, 1)$ are equal:

$$2x + b \Big|_{x=1} = c - 2x \Big|_{x=1}$$

$$2 + b = c - 2 \Rightarrow 2 + b = 2 - 2$$

$$\Rightarrow b = -2$$

$$\Rightarrow a = 2$$

$$\begin{aligned} \text{So } a + 2(b) + c &= 2 - 4 + 2 \\ &= 0 \end{aligned}$$

28. If α is a positive real number and

$$\lim_{x \rightarrow 0} \frac{\sin^\alpha(8x)}{\ln^\alpha(1+4x)} = 64,$$

then $\alpha =$

$$\lim_{x \rightarrow 0} \left(\frac{\sin(8x)}{\ln(1+4x)} \right)^\alpha = 64 \quad (*)$$

(a) 6

(b) 8

(c) 4

(d) 7

(e) 9

$$\text{As } \lim_{x \rightarrow 0} \frac{\sin(8x)}{\ln(1+4x)} \stackrel{L'H}{=} \lim_{x \rightarrow 0} \frac{8 \cos(8x)}{\frac{4}{1+4x}} = \frac{8 \cdot 1}{4} = 2$$

then from $(*)$, we get

$$2^\alpha = 64$$

$$\text{So } \alpha = 6$$